Probabilistic functions and cryptographic oracles in higher-order logic

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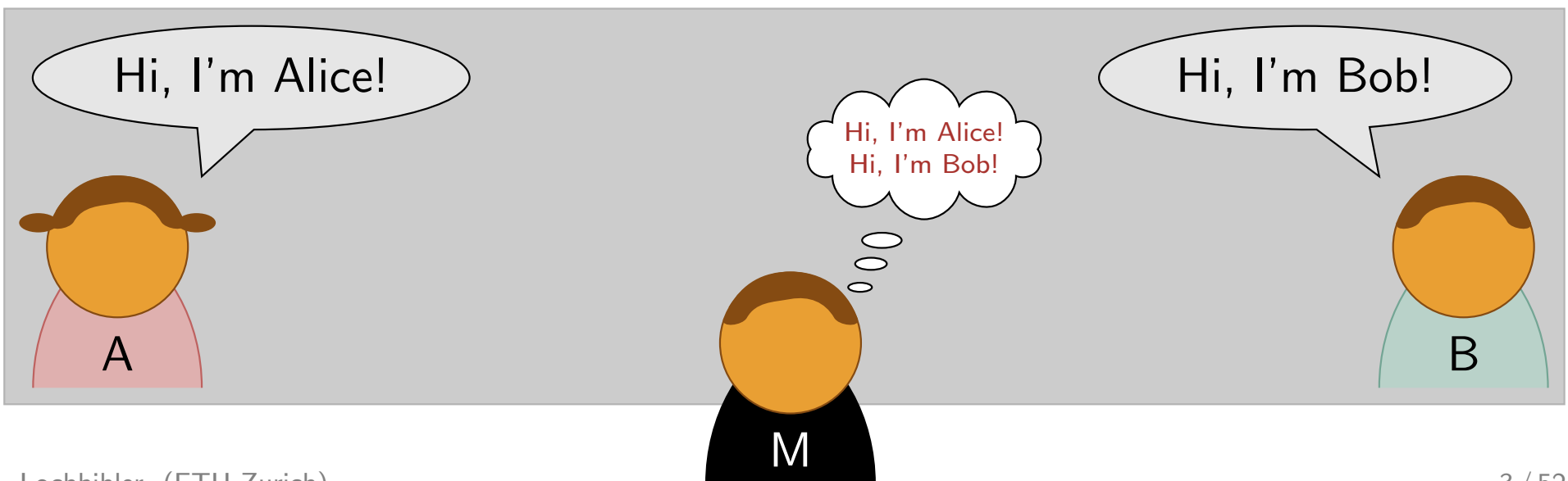
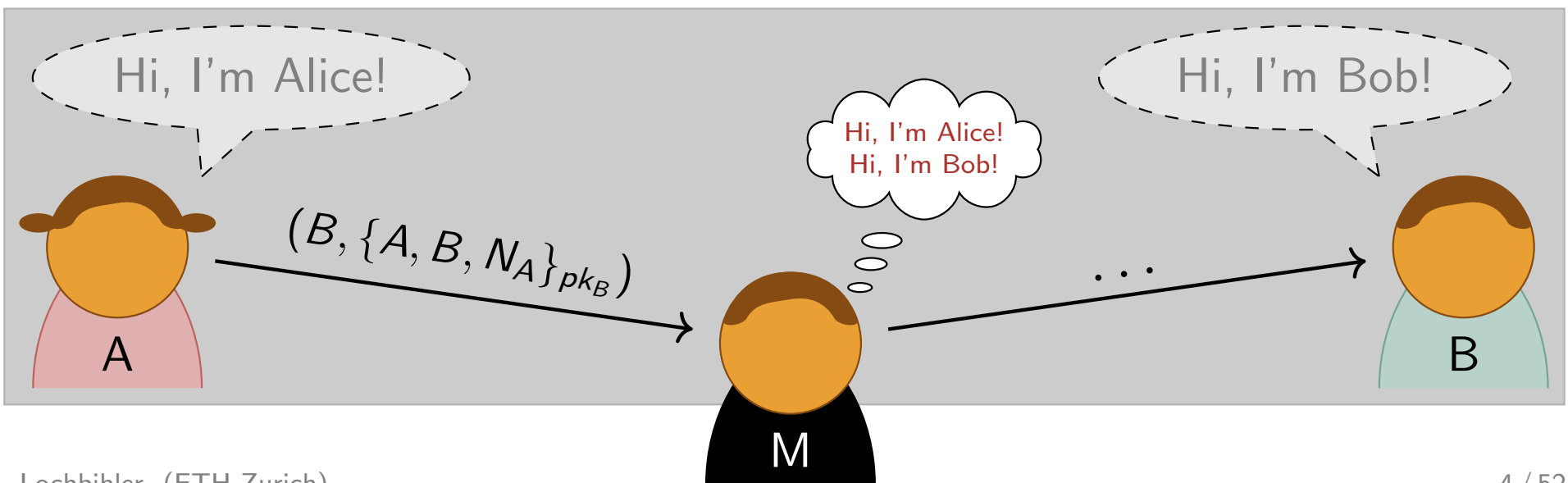
Institute of Information Security

ETH zürich


Hi, I'm Alice!
Hi, I'm Bob!
A
B


Hi, I'm Alice!
Hi, I'm Alice!
Hi, I'm Bob!
Hi, I'm Bob!
A
M
B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$(B, \{A, B, N_A\}_{pk_B})$
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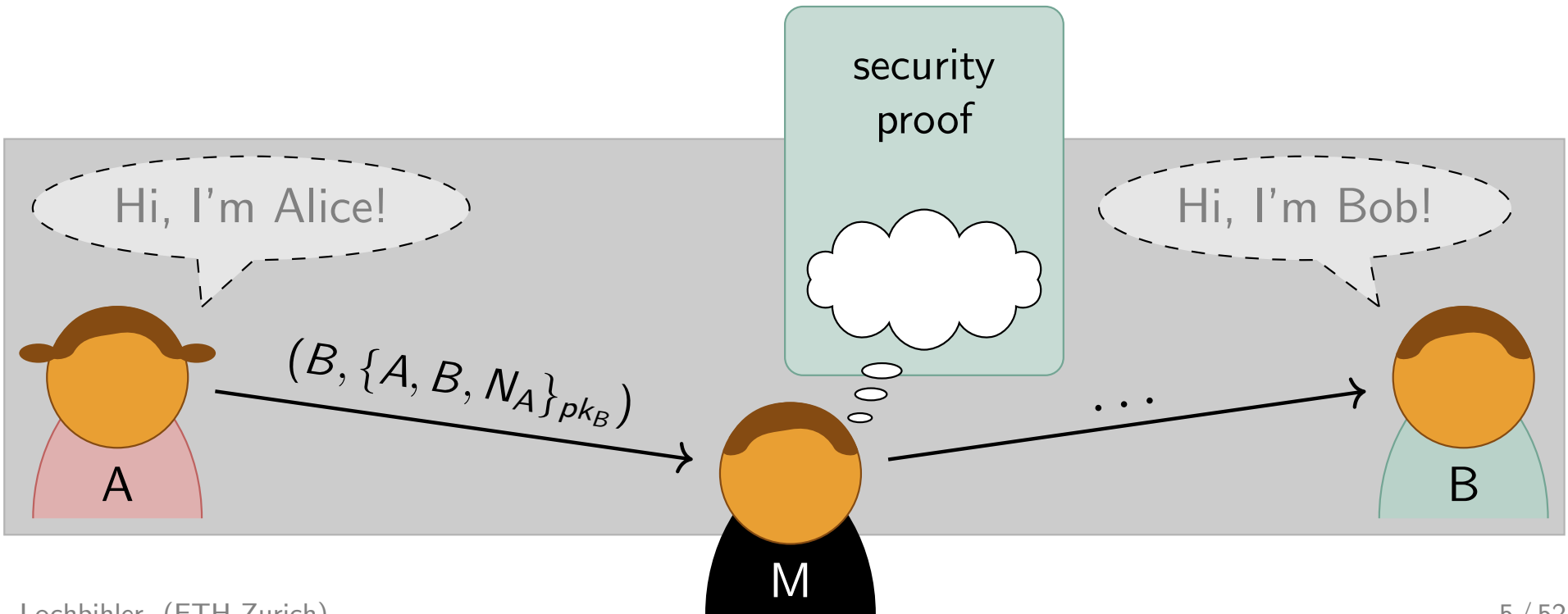
security
proof
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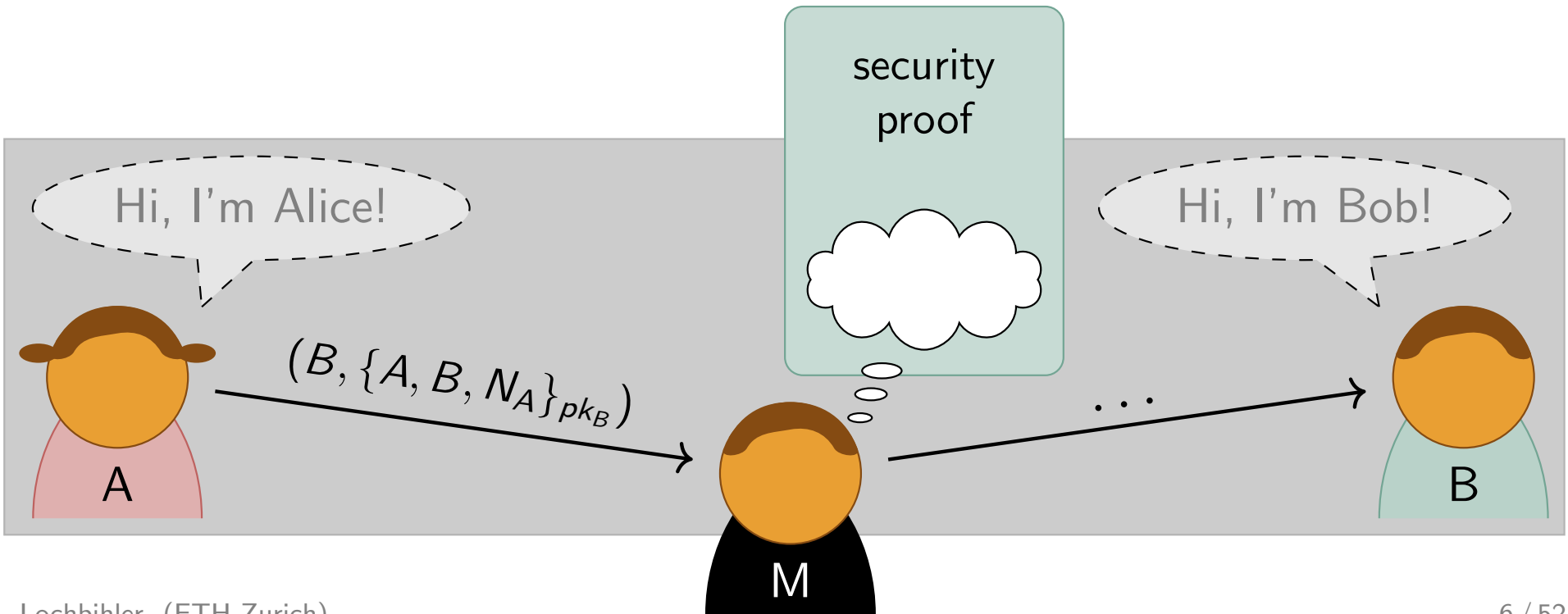
adversary model	Dolev-Yao
cryptography	perfect
reasoning power	unbounded
messages	$\{\langle B, N_A \rangle\}_k$
properties	reachability

ProVerif, OFMC, Scyther, Tamarin



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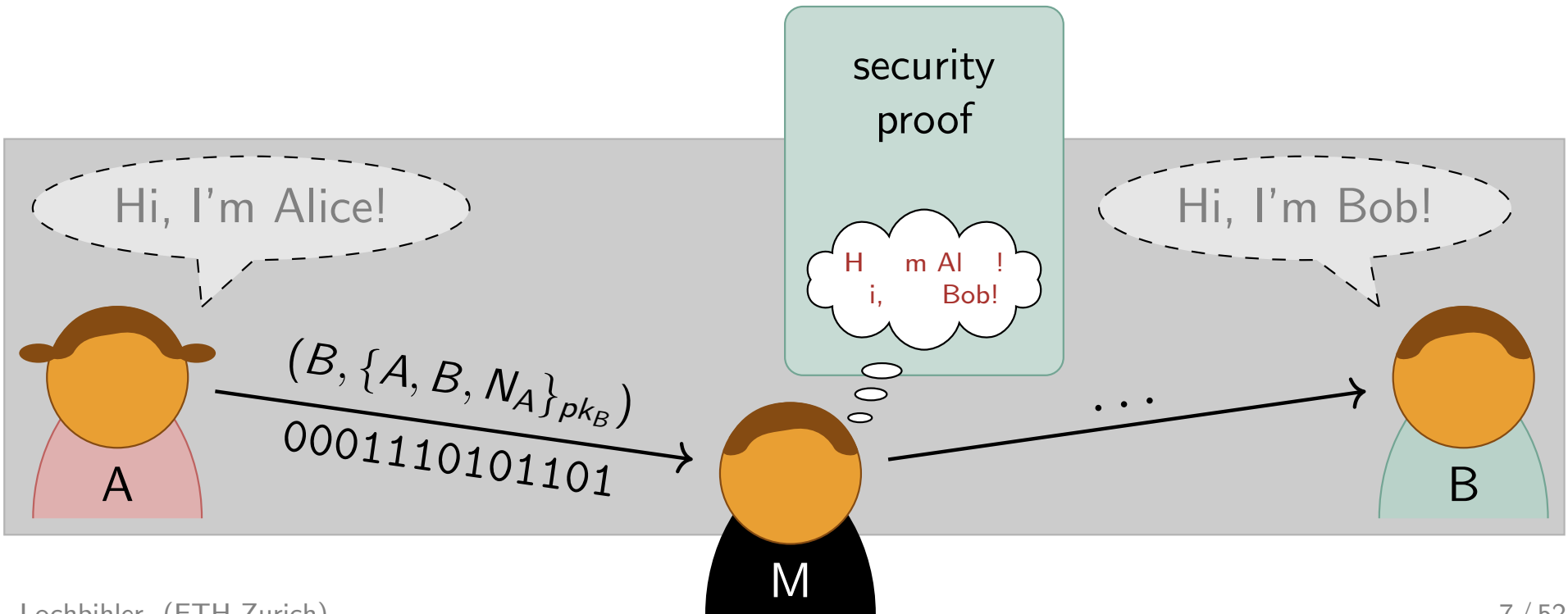

adversary model	Dolev-Yao	computational
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reasoning power	unbounded	efficient
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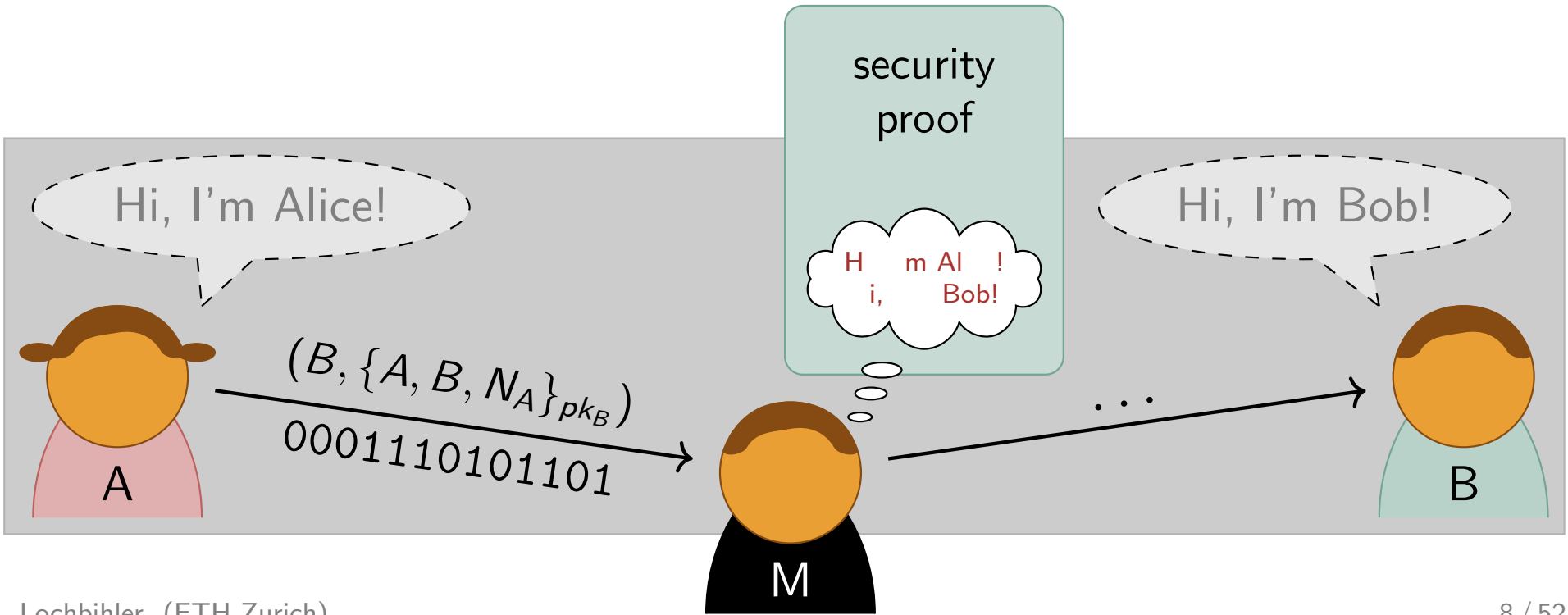

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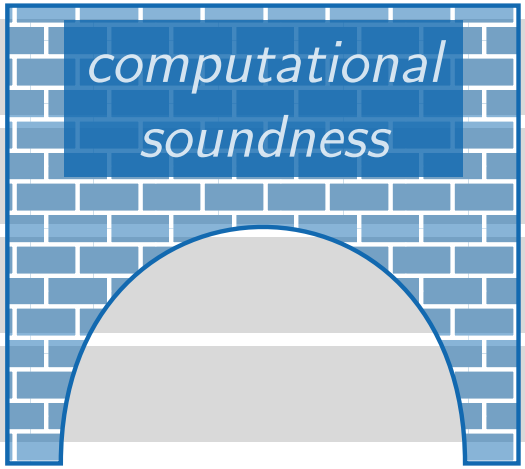

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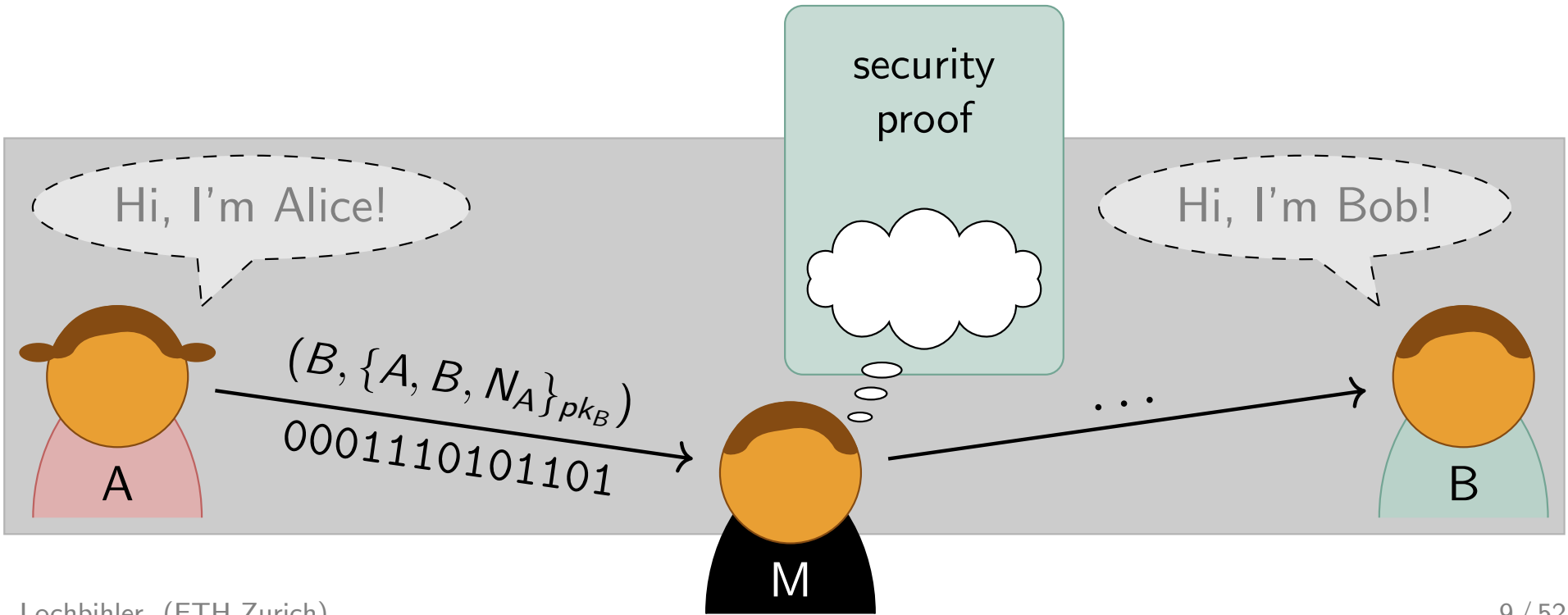

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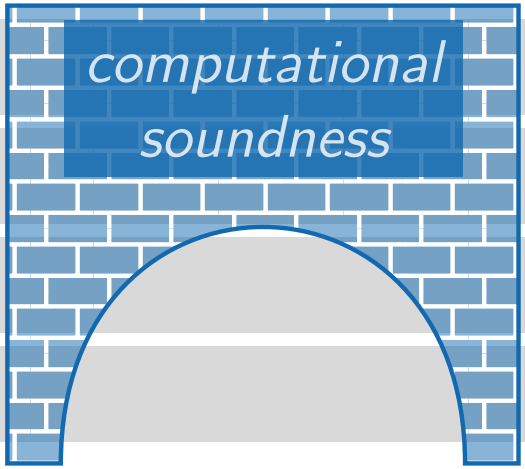

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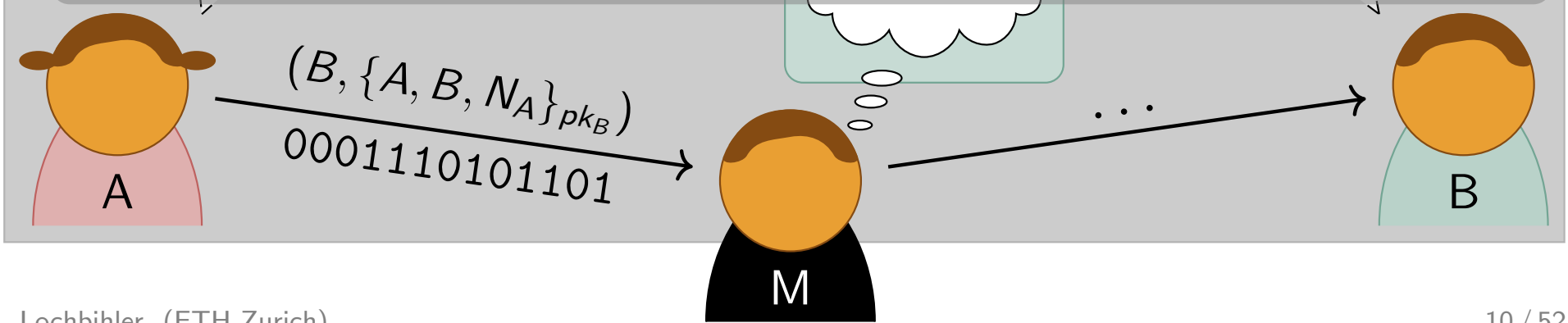
Goal: Obtain computational guarantees for Dolev-Yao security proofs by formalising a computational soundness proof.

This talk: **A framework for formalising computational arguments**



Frameworks for formalising computational arguments

	embedding	symbolic messages	proof automation	trusted base
CertiCrypt	deep	⊖	⊖	⊕ Coq
EasyCrypt	axiomatic	⊖	⊕	⊖ EasyCrypt + SMT
Verypto	deep	0	⊖	0 Isabelle + axioms
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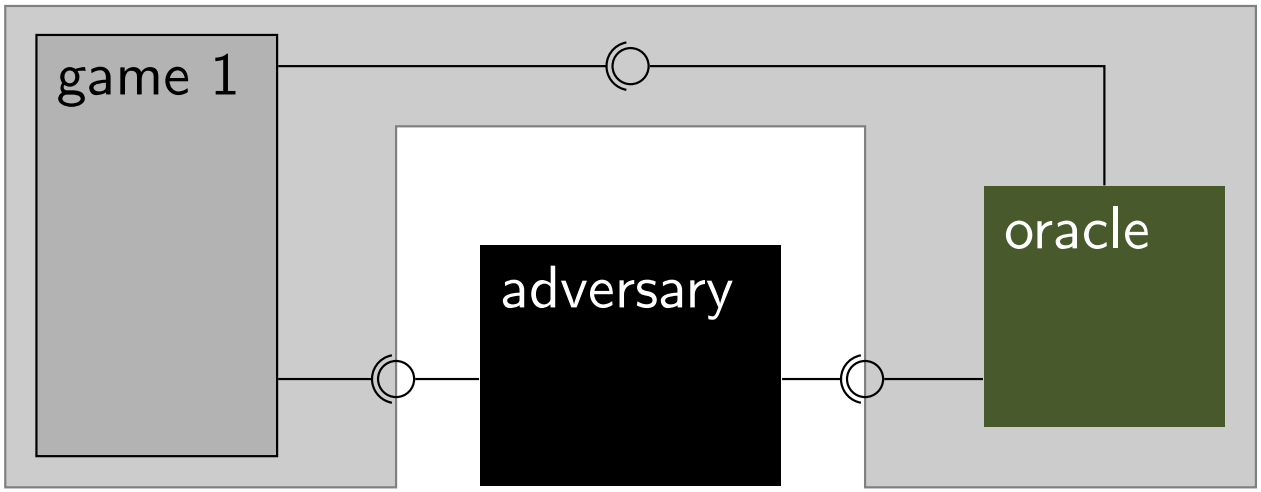
Overview

1. Probabilistic language in higher-order logic (HOL)
   - new semantic domain
   - shallow embedding
   - oracle access, monadic sequencing, exceptions, recursion
2. Systematic way to *find* reasoning rules for language primitives
3. Textbook examples formalised in Isabelle/HOL

Structure of computational arguments



Security:

Probability of winning game 1 is small for any efficient adversary.

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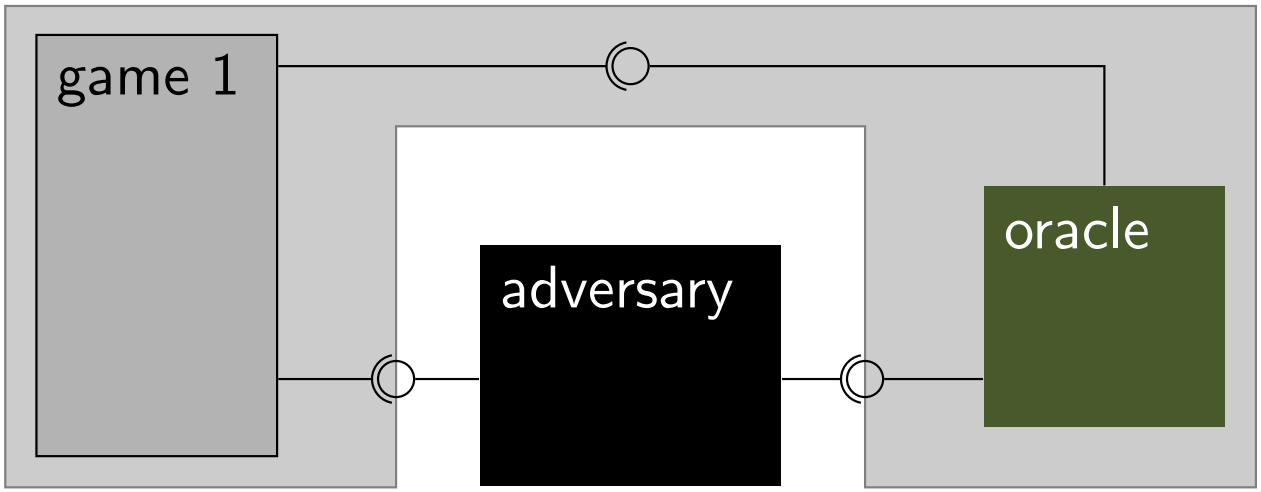

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  assert (valid-plain$(m_0) \wedge$ valid-plain(m_1));
  $c^* \leftarrow$ aenc$(pk,$ **if** b **then** m_0 **else** $m_1)$;
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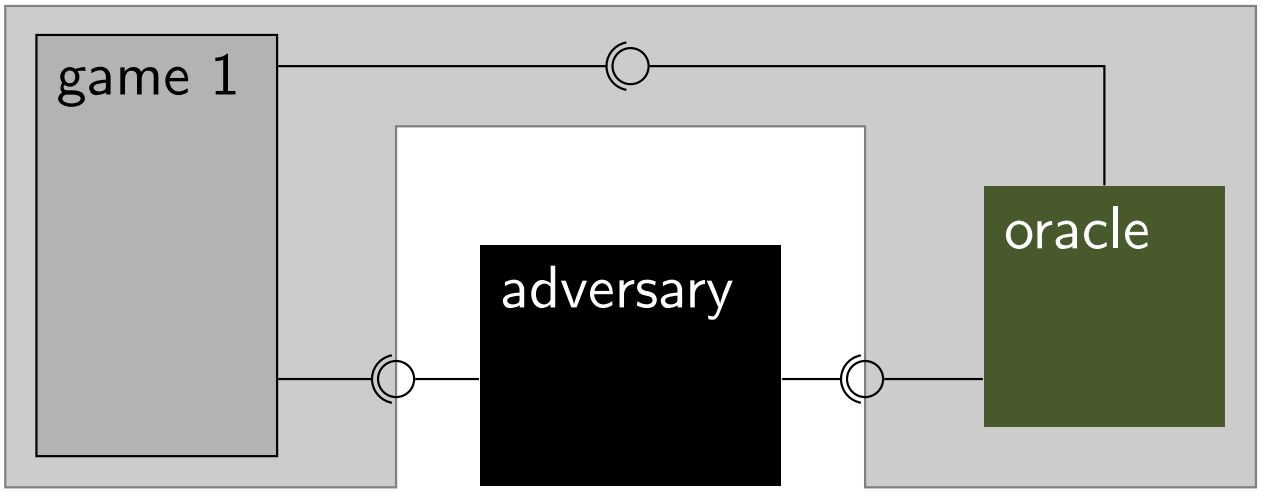

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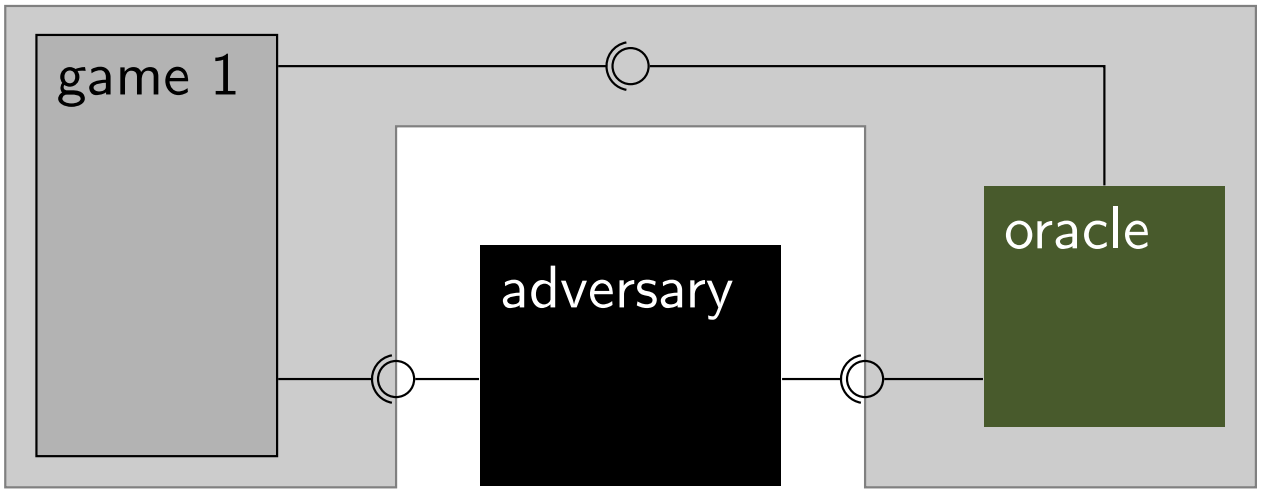
monad

Structure of computational arguments



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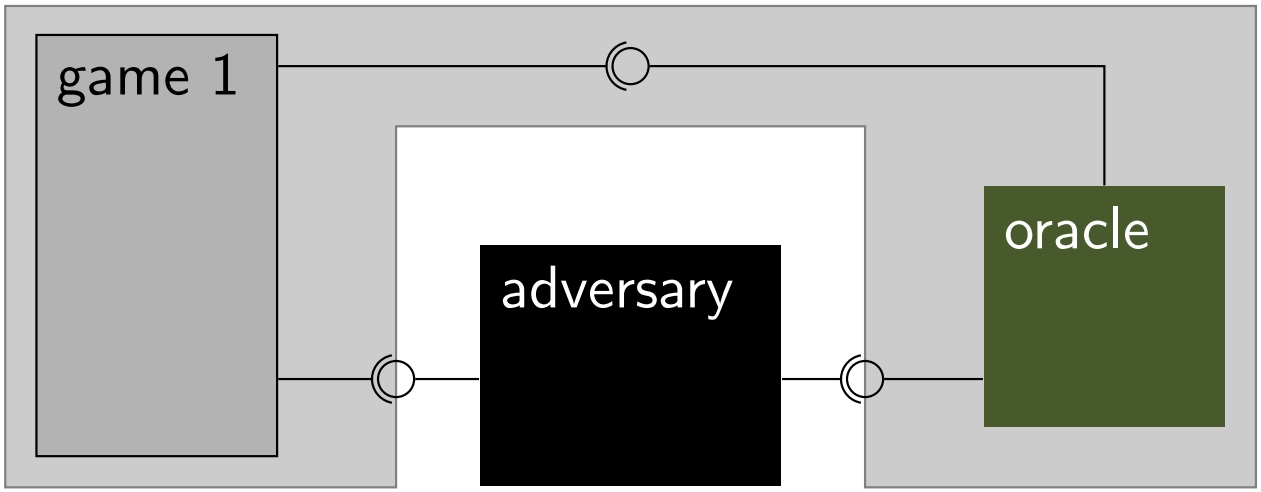
monad

sampling

Structure of computational arguments



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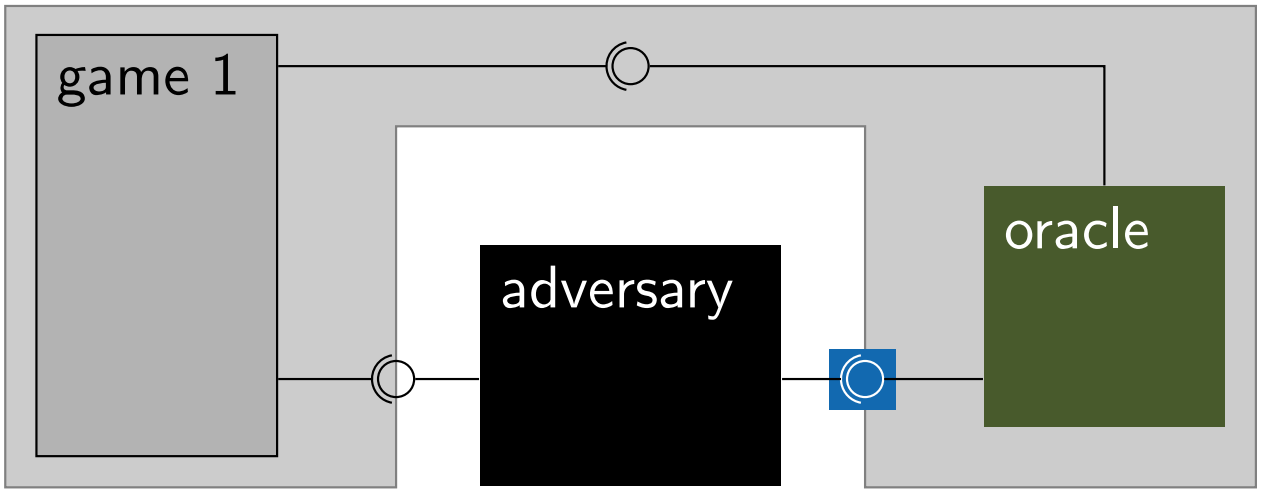
monad

sampling

assertions and error handling

Structure of computational arguments



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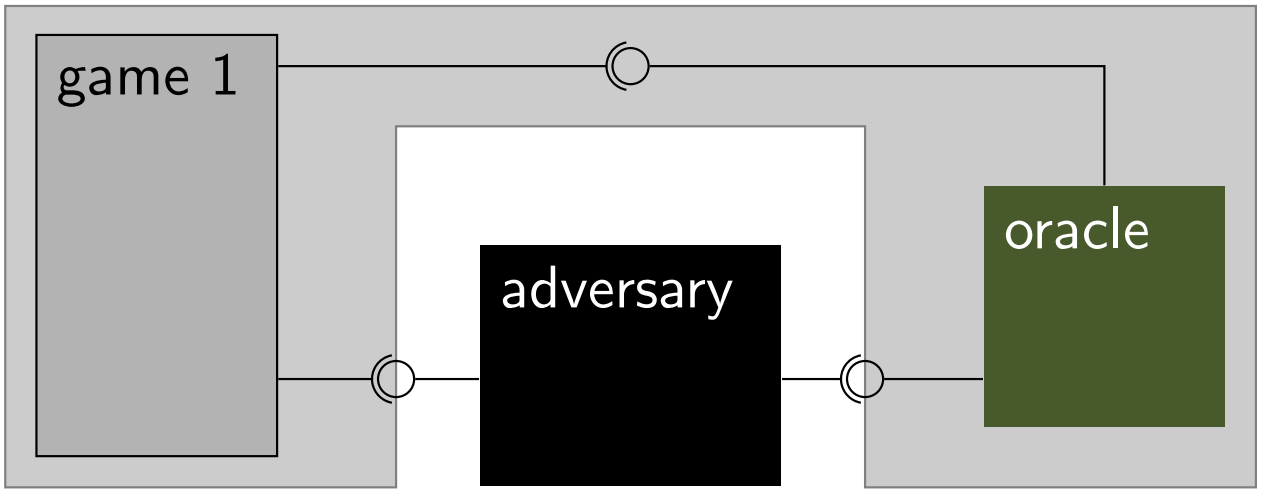
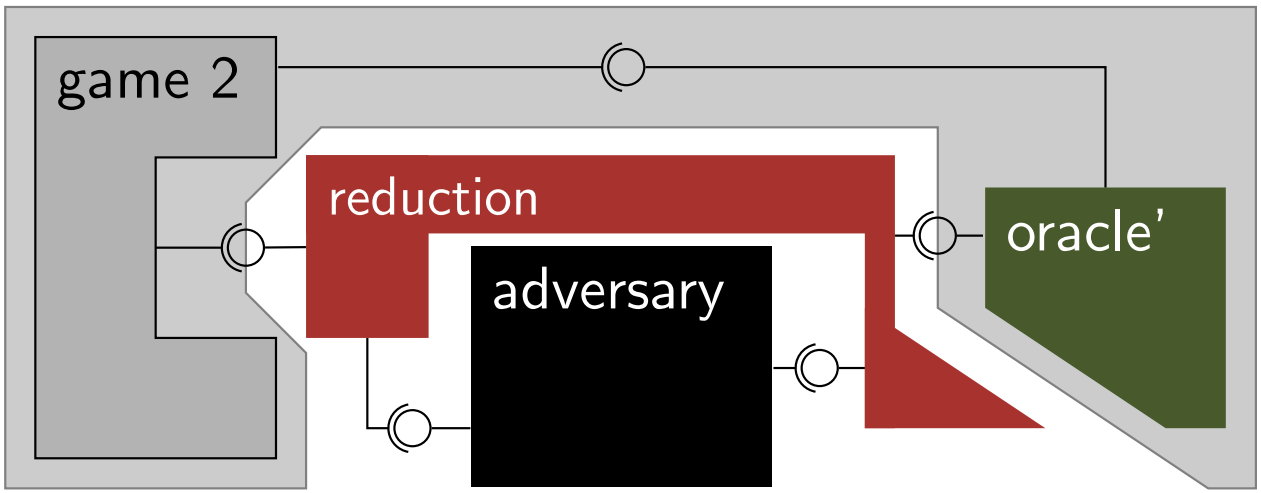


reduction



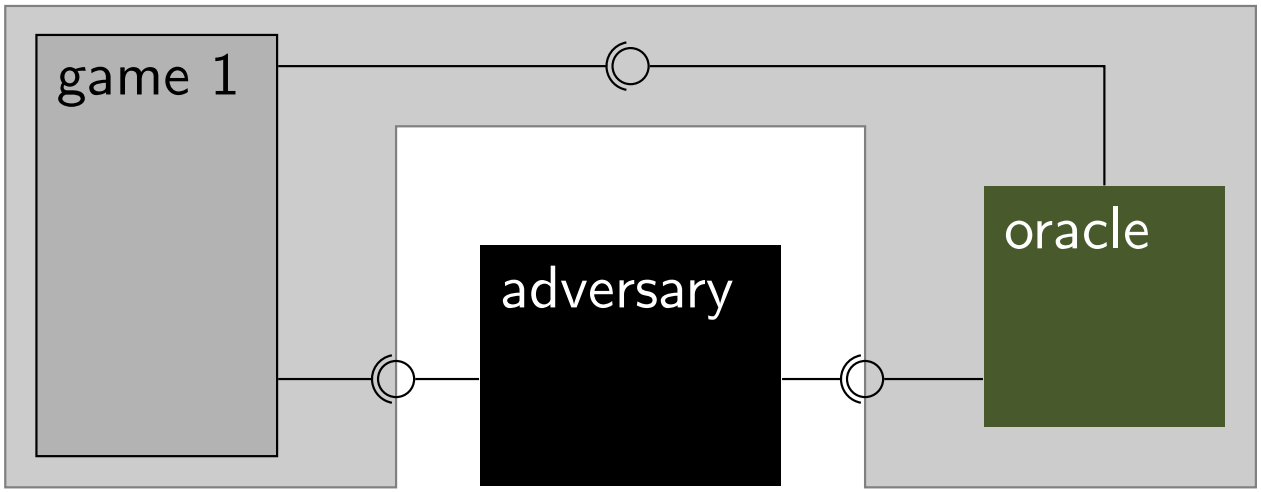
Security:

Probability of winning game 1 is small for any efficient adversary.

Assumption:

Probability of winning game 2 is small for any efficient adversary.

Structure of computational arguments



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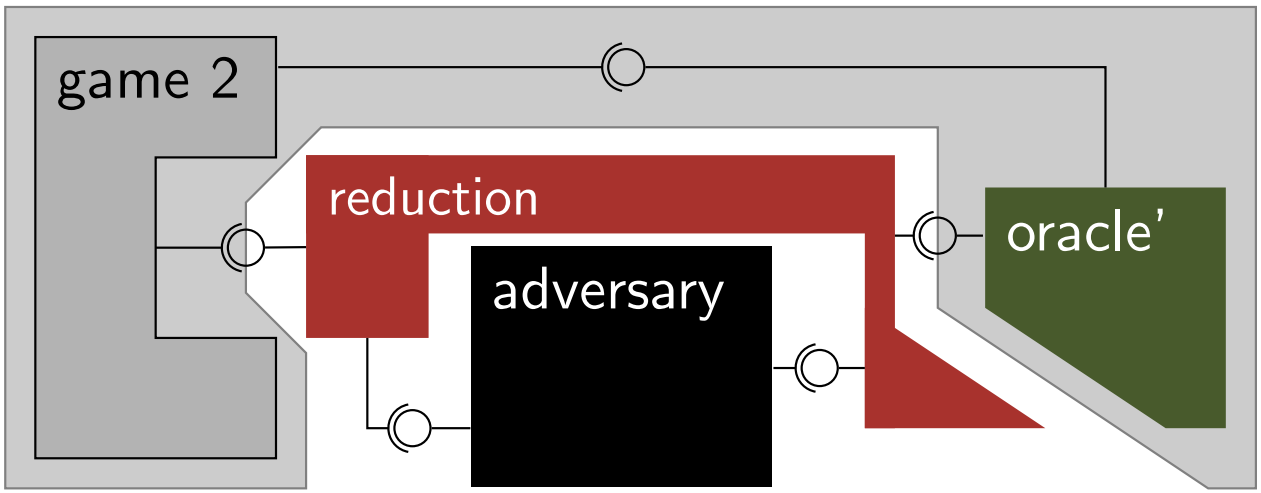

Security theorem:

Probability of winning game 1 is small for any efficient adversary.

⇑

Probability of winning game 2 is small for any efficient adversary.

Discrete subprobabilities as semantic domain

typedef α sprob $= \left\{ f : \alpha \to \mathbb{R}_0^+ \mid \sum_x f(x) \leq 1 \right\}$

Theorem: α sprob is a chain-complete partial order.

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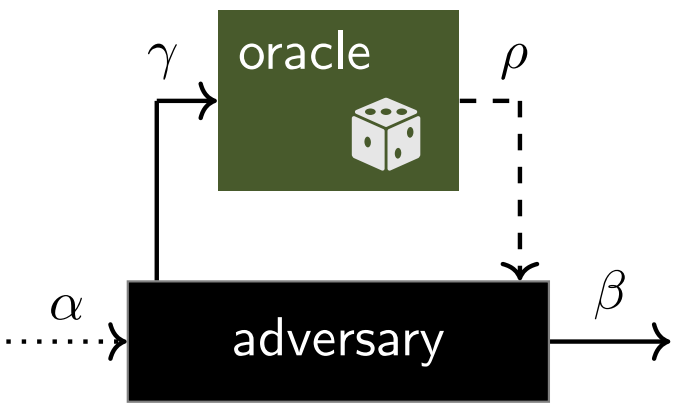

$\gamma \to \rho$ sprob

$\alpha \to \beta$ sprob

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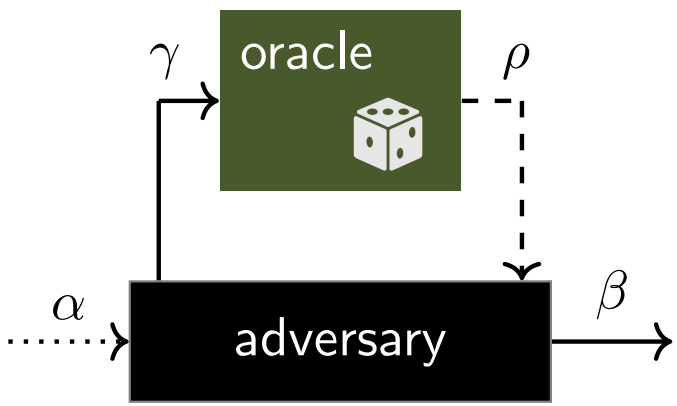

$$(\gamma \to \rho \text{ sprob}) \to \alpha \to \beta \text{ sprob}$$

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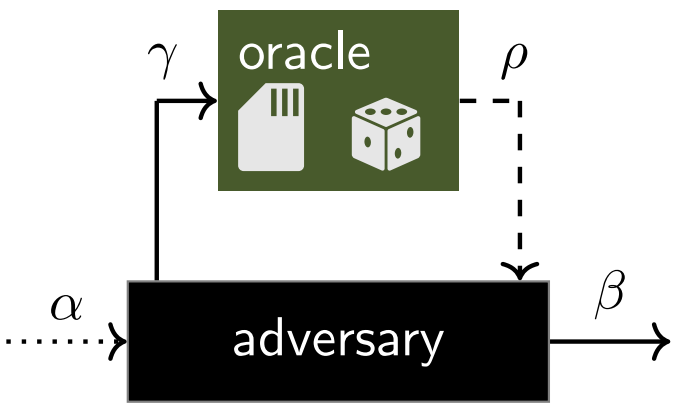

$\sigma \to \gamma \to (\rho \times \sigma)$ sprob

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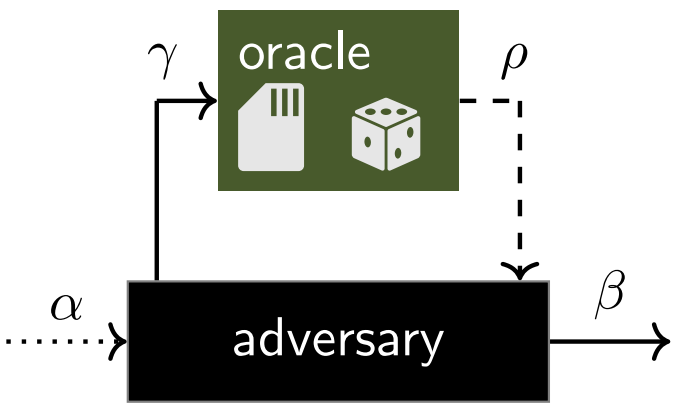

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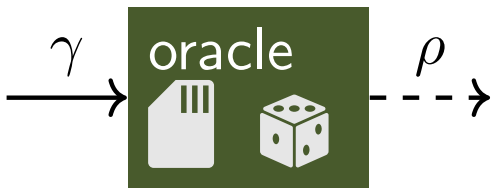
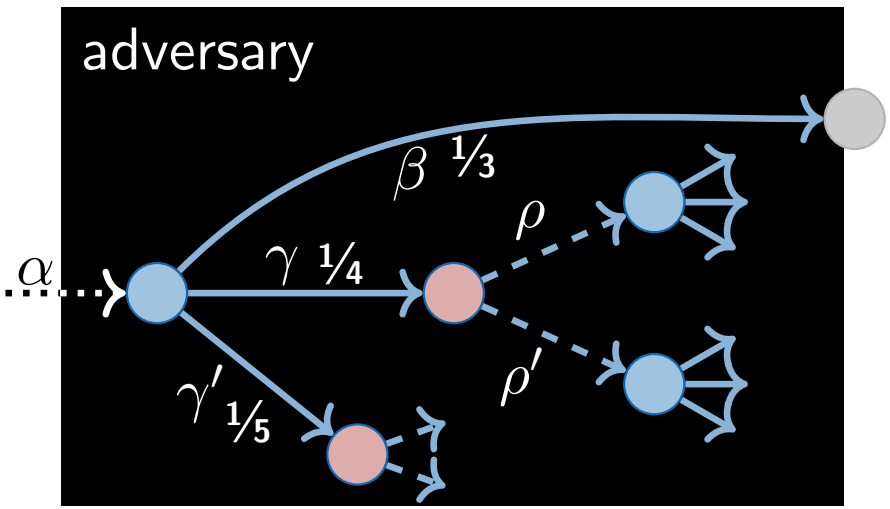
Generative probabilistic values

Second attempt to model oracle access:



oracle: $\sigma \to \gamma \to (\rho \times \sigma)$ sprob



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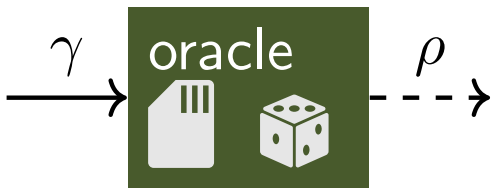

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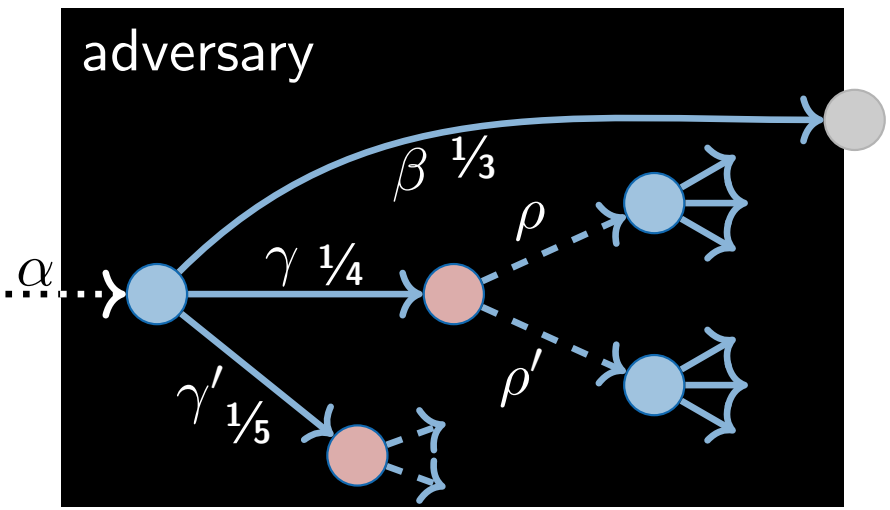

adversary: $\alpha \to$ gpv

gpv $\cong$ (β $+ \gamma \times$ rpv) sprob
rpv $\cong \rho \to$ gpv

codatatype gpv =
Gpv (($\beta + \gamma \times (\rho \Rightarrow$ gpv) sprob)

Express operators for sequencing, failure, composition, ...

Generative probabilistic values

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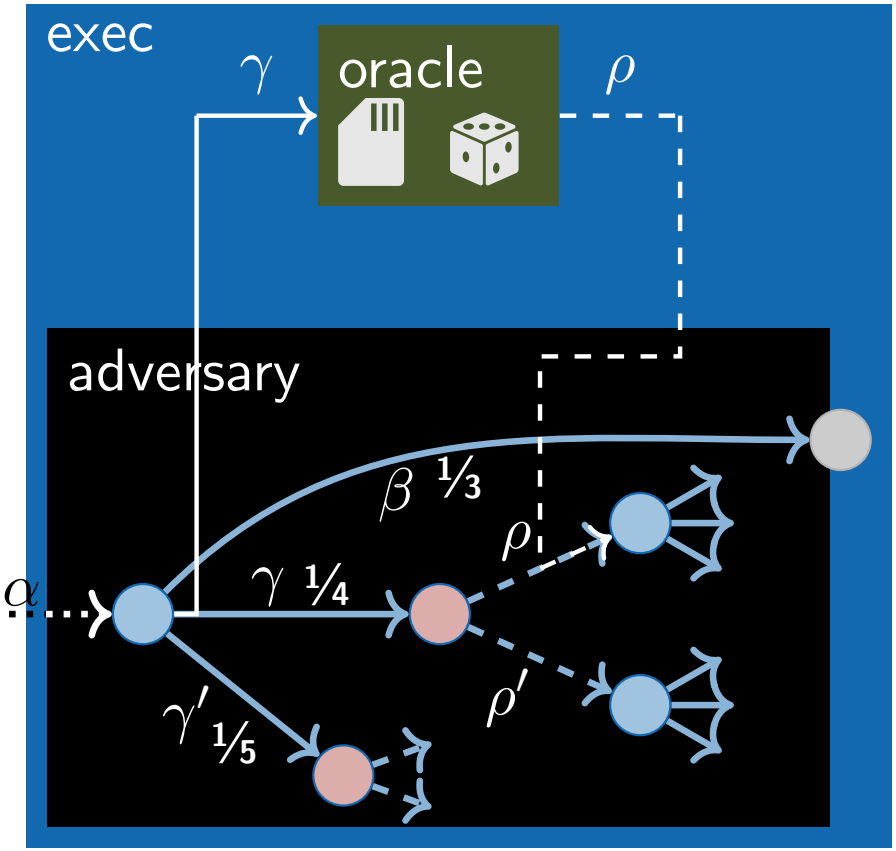

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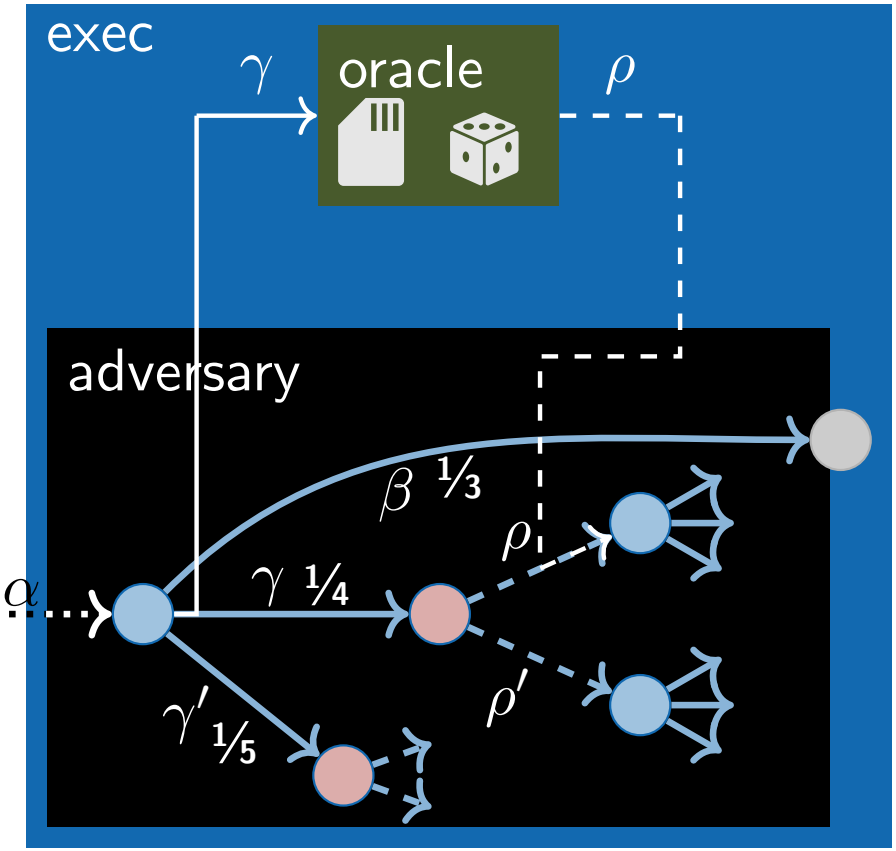

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Express operators for sequencing, failure, composition, ...

gpv also used for reductions and games

Rules for reasoning about game transformations

Equational reasoning

- Shallow embedding supports many equalities
- Example: commutativity for sprob

$$\begin{array}{l}\textbf{do}\ \{ \\ \quad x \leftarrow ppp; \\ \quad y \leftarrow q; \\ \quad f(x,y)\ \}\end{array} \quad = \quad \begin{array}{l}\textbf{do}\ \{ \\ \quad y \leftarrow q; \\ \quad x \leftarrow ppp; \\ \quad f(x,y)\ \}\end{array}$$

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Relational reasoning

- Lift relation A on outcomes to relation $\uparrow A \uparrow$ on sprobs or gpvs
- Example: sequencing

$$\frac{p \uparrow A \uparrow q \qquad \forall (x,y) \in A.\ f(x) \uparrow B \uparrow g(y)}{(\textbf{do}\ \{x \leftarrow p; f(x)\ \}) \uparrow B \uparrow (\textbf{do}\ \{y \leftarrow q; g(y)\ \})}$$

Finding relational rules

Shape of rule determined by operator type

$$\begin{array}{rlllllll} \mathsf{bind}_{\mathsf{sprob}} : & \alpha\ \mathsf{sprob} & \Rightarrow & (\ \alpha & \Rightarrow & \beta\ \mathsf{sprob}\ ) & \Rightarrow & \beta\ \mathsf{sprob} \\ & \uparrow A \uparrow_{\mathsf{sprob}} & \Mapsto & (\ A & \Mapsto & \uparrow B \uparrow_{\mathsf{sprob}}\ ) & \Mapsto & \uparrow B \uparrow_{\mathsf{sprob}} \end{array}$$

Replace types by relations

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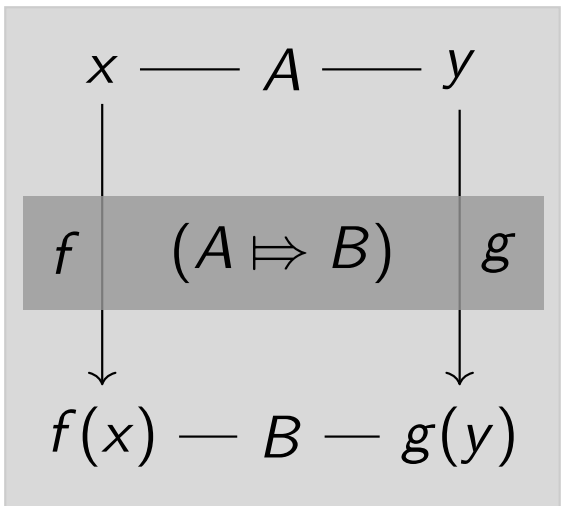
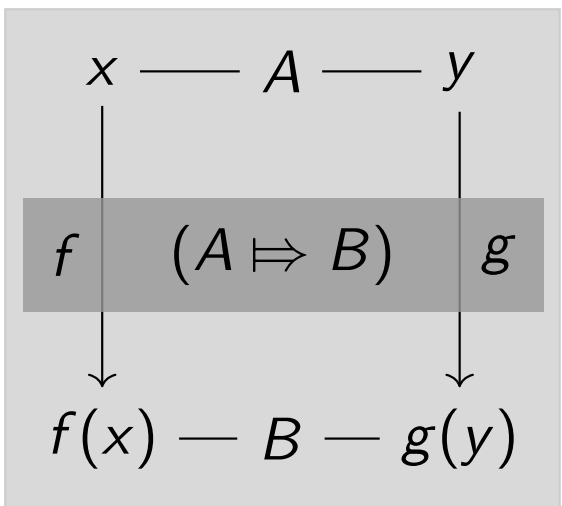


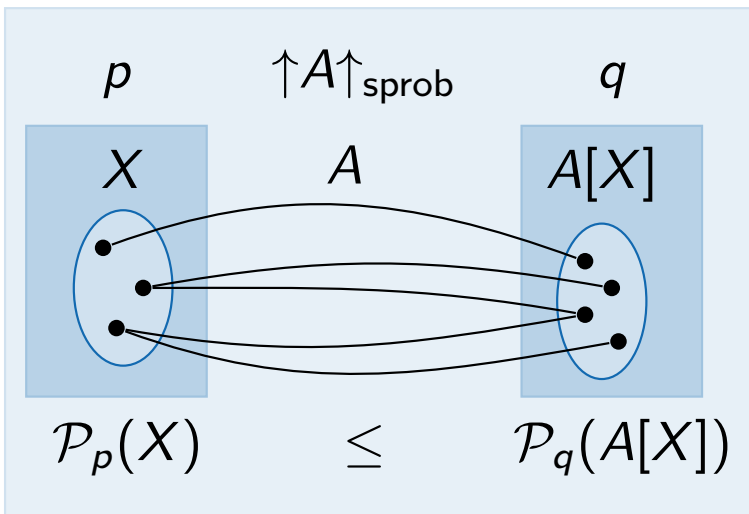
Finding relational rules

Shape of rule determined by operator type

$$\begin{array}{rcccccccc} \mathsf{bind}_{\mathsf{sprob}} : & \alpha\ \mathsf{sprob} & \Rightarrow & (\ \alpha & \Rightarrow & \beta\ \mathsf{sprob}\ ) & \Rightarrow & \beta\ \mathsf{sprob} \\ & \uparrow A \uparrow_{\mathsf{sprob}} & \Mapsto & (\ A & \Mapsto & \uparrow B \uparrow_{\mathsf{sprob}}\ ) & \Mapsto & \uparrow B \uparrow_{\mathsf{sprob}} \end{array}$$

Replace types by relations





Finding relational rules

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$$\mathsf{bind}_{\mathsf{sprob}} : \alpha\ \mathsf{sprob} \Rightarrow (\alpha \Rightarrow \beta\ \mathsf{sprob}) \Rightarrow \beta\ \mathsf{sprob}$$

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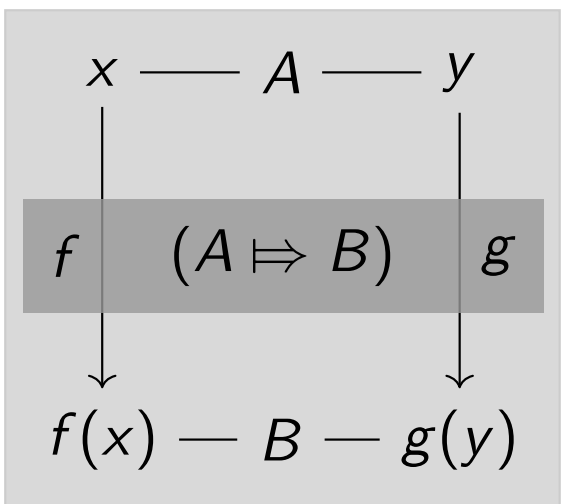
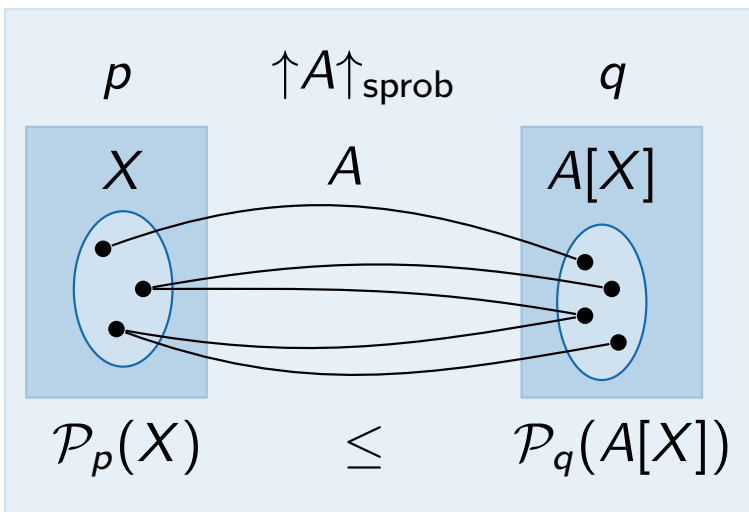
Replace types by relations (Reynolds: relational parametricity)





Finding relational rules

Shape of rule determined by operator type

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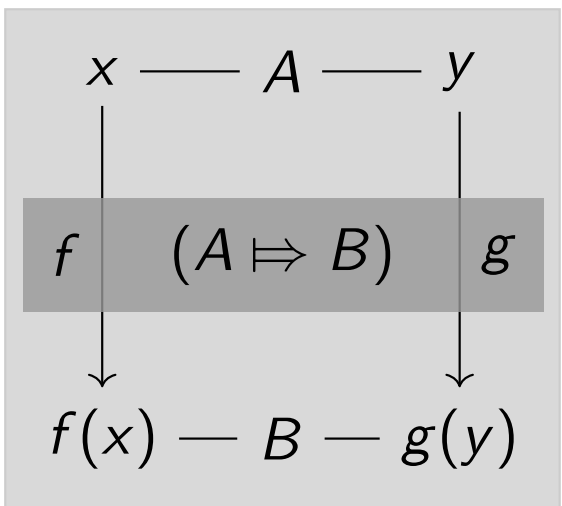
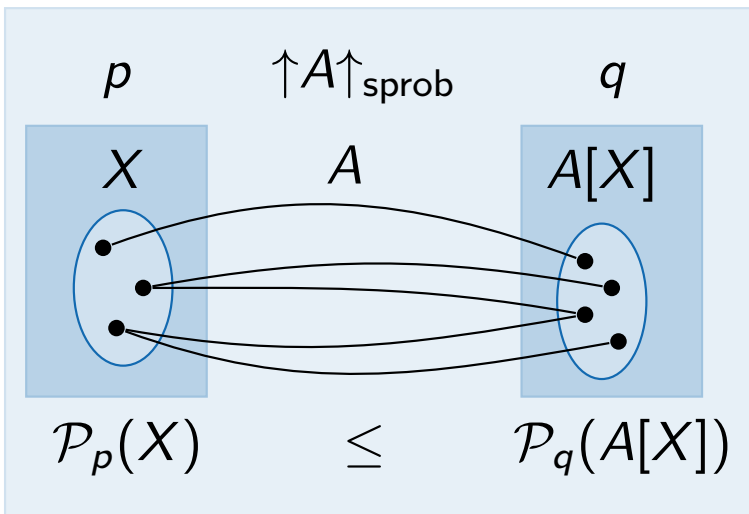
Replace types by relations (Reynolds: relational parametricity)

$$\frac{p \uparrow A \uparrow_{\mathsf{sprob}} q \qquad \forall (x,y) \in A.\ f(x) \uparrow B \uparrow_{\mathsf{sprob}} g(y)}{(\mathbf{do}\ \{x \leftarrow p; f(x)\}) \uparrow B \uparrow_{\mathsf{sprob}} (\mathbf{do}\ \{y \leftarrow q; g(y)\})}$$





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Used this approach to find rules for new operators, e.g.

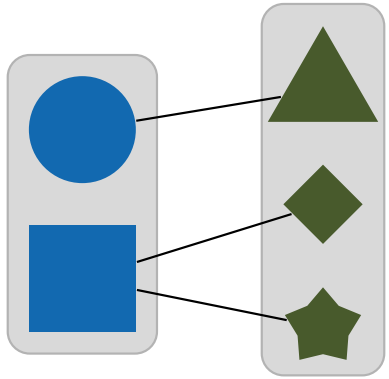
$$\frac{\mathcal{A}_1 \uparrow A\uparrow_{\mathsf{gpv}} \mathcal{A}_2 \qquad \mathcal{O}_1\ (S \Rightarrow (=) \Rightarrow \uparrow(=) \times S\uparrow_{\mathsf{sprob}})\ \mathcal{O}_2 \qquad \sigma_1\ S\ \sigma_2}{\mathsf{exec}(\mathcal{A}_1, \mathcal{O}_1, \sigma_1) \uparrow A \times S\uparrow_{\mathsf{sprob}} \mathsf{exec}(\mathcal{A}_2, \mathcal{O}_2, \sigma_2)}$$

Sampling is conditionally parametric

$$\mathcal{P}_{\text{uniform}\ \Omega}(X) = \frac{|X|}{|\Omega|}$$

uniform :

α set	$\Rightarrow$	α sprob
$\uparrow A \uparrow_{\text{set}}$	$\mapsto$	$\uparrow A \uparrow_{\text{sprob}}$



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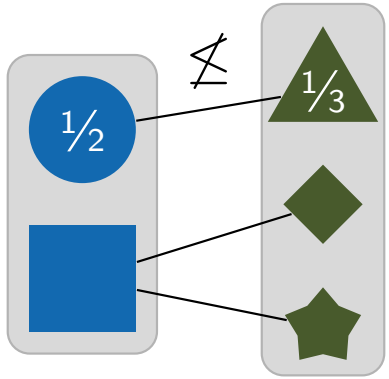

Wadler, Reynolds: Polymorphic equality is not parametric!

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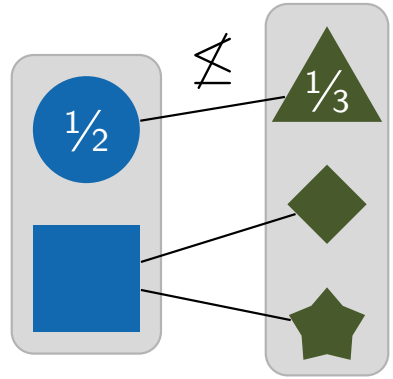


Wadler, Reynolds: Polymorphic equality is not parametric!

A must respect equality!

$$\frac{(=)\ (A \Mapsto A \Mapsto \text{rel}_{\text{bool}})\ (=)}{\text{uniform}\ (\uparrow A \uparrow_{\text{set}} \Mapsto \uparrow A \uparrow_{\text{sprob}})\ \text{uniform}}$$

$$\frac{A \text{ is bijection between } \Omega_1 \text{ and } \Omega_2}{\text{uniform } \Omega_1 \ \ \uparrow A \uparrow_{\text{sprob}} \ \ \text{uniform } \Omega_2}$$

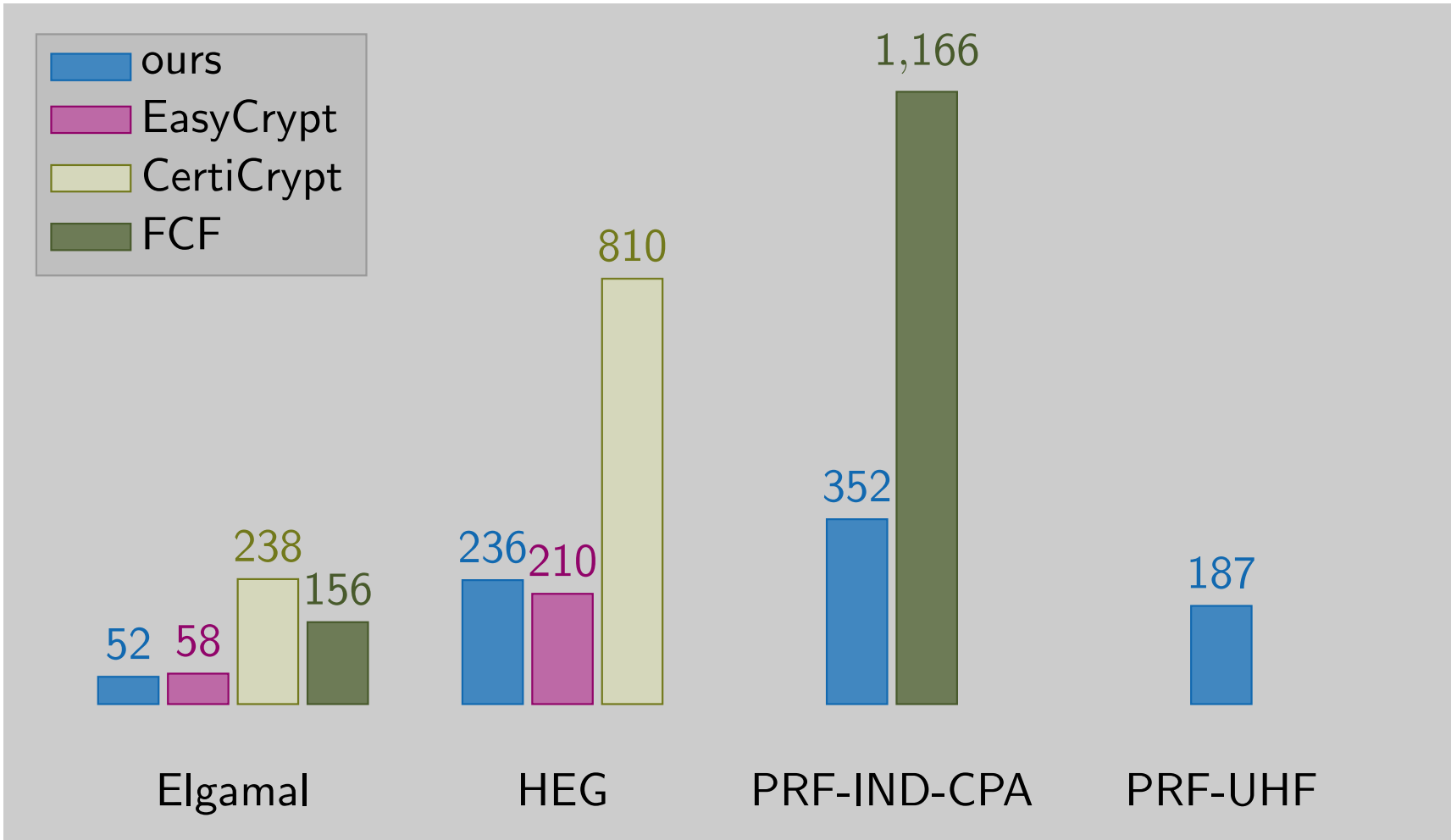
Special case: **one-time pad**

$$\text{map}_{\text{sprob}}\ (\text{xor}\ m)\ (\text{uniform}\ \{0,1\}^n) = \text{uniform}\ \{0,1\}^n$$

Applications

Used framework to verify textbook cryptographic constructions.

Line counts of proof of concrete security theorem



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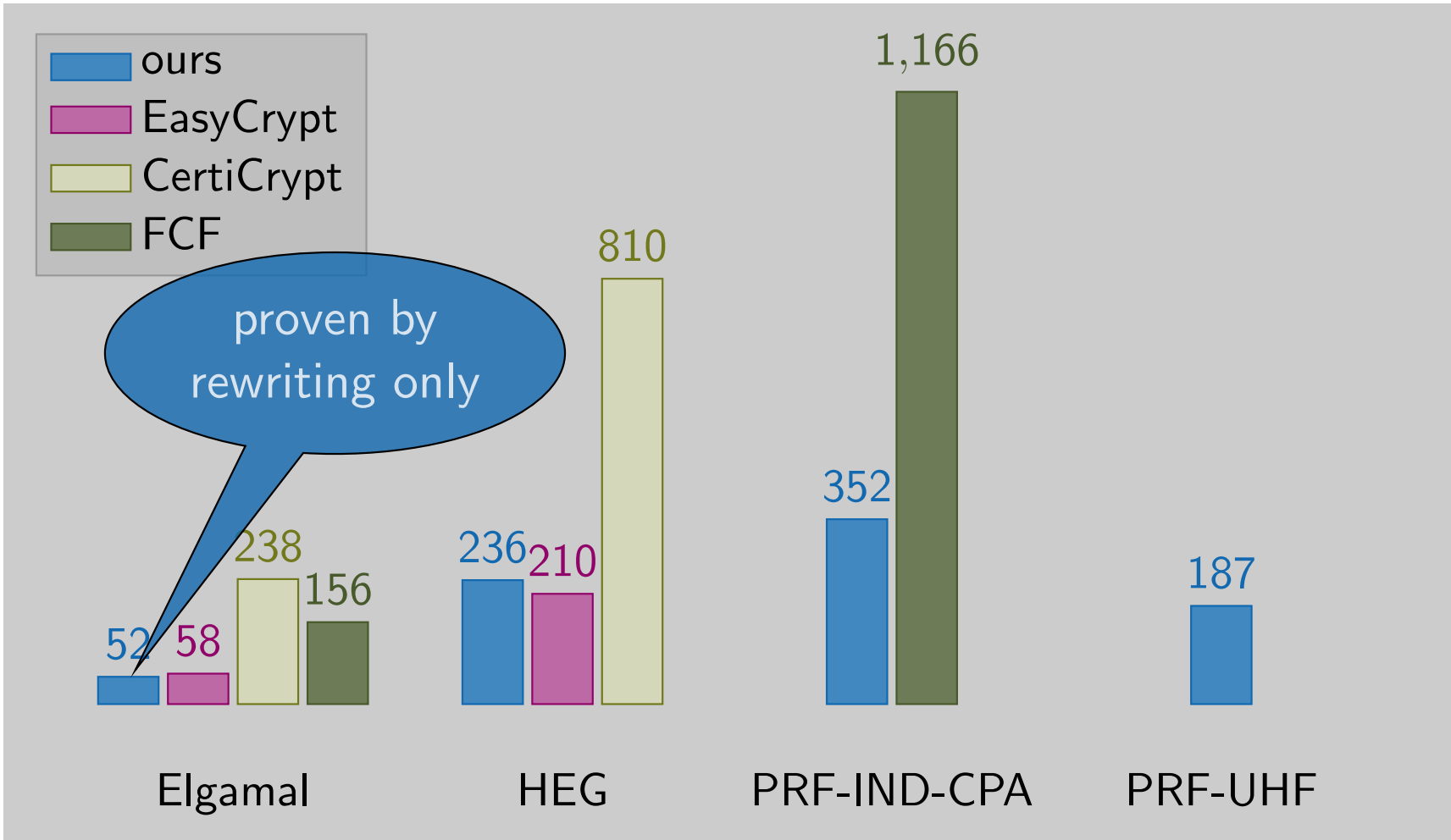

```
locale elgamal = elgamal_base + cyclic_group 𝒢 +
  assumes finite_group: "finite (carrier 𝒢)"
begin

lemma advantage_elgamal: "ddh.advantage (elgamal_adversary 𝒜) = ind_cpa.advantage 𝒜"
proof -
  obtain 𝒜1 𝒜2 where [simp]: "𝒜 = (𝒜1, 𝒜2)" by(cases 𝒜)
  note [simp] = order_gt_0_iff_finite finite_group
  have "ddh.ddh_1 (elgamal_adversary 𝒜) = TRY do {
        let q = order 𝒢;
        x ← sample_uniform q;
        y ← sample_uniform q;
        ((msg1, msg2), σ) ← 𝒜1 (g (^) x);
        _ :: unit ← assert_spmf (msg1 ∈ carrier 𝒢 ∧ msg2 ∈ carrier 𝒢);
        b ← coin_spmf;
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      } ELSE coin_spmf"
    apply(simp add: Let_def try_spmf_bind_out split_def bind_map_spmf o_def ddh.ddh_1_def)
    apply(simp add: bind_commute_spmf[where p=coin_spmf] bind_commute_spmf[where q="𝒜1 _"])
    apply(simp add: bind_commute_spmf[where q="assert_spmf _"])
    done
  also have "... = TRY do {
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```
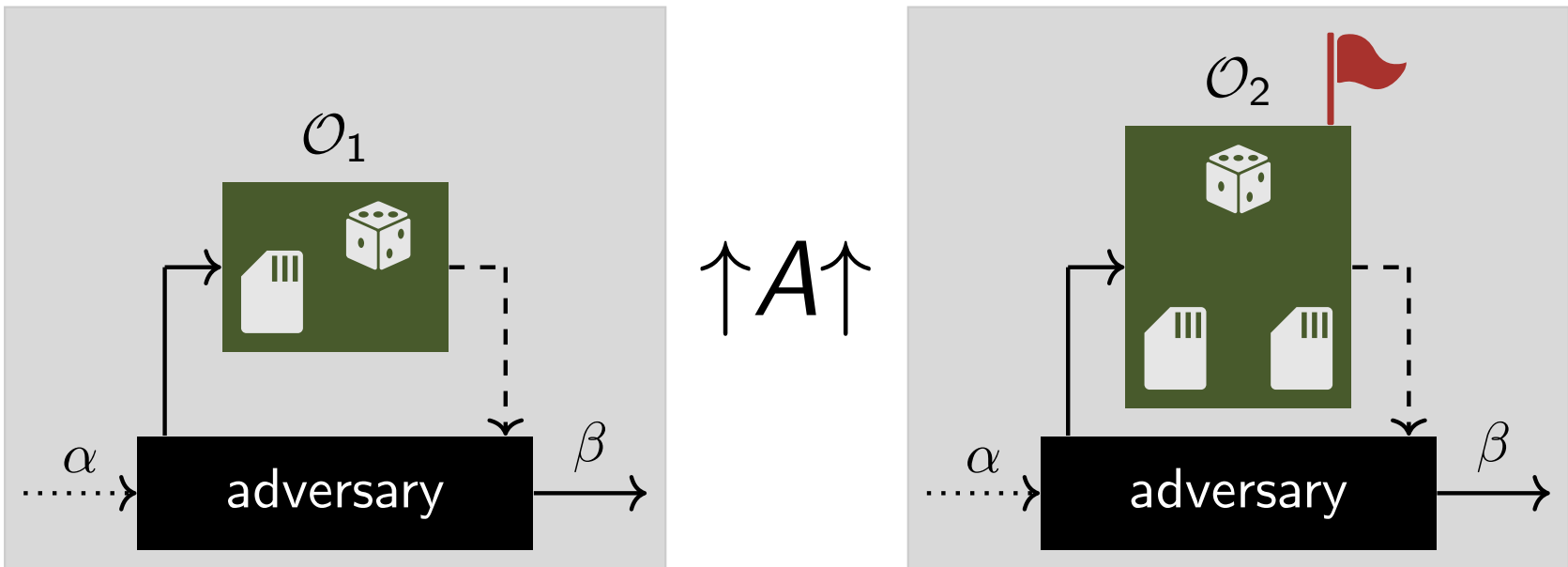

Parametricity and representation independence

- composition preserves parametricity
- exploit theory on relational parametricity

Example: representation independence [Mitchell]



- Prove bisimulation for S: $\mathcal{O}_1 \ (S \Mapsto (=) \Mapsto \uparrow(=) \times S\uparrow) \ \mathcal{O}_2$
- Lift to whole game using parametricity

Random oracle with map $s : \alpha \rightharpoonup \{0,1\}^n$

$$
\begin{array}{l}
\mathsf{RO}(s, x) = \\
\quad \textbf{if } x \in \mathsf{dom}(s) \textbf{ then} \\
\qquad \mathsf{return}_{\mathsf{sprob}}\ (s(x), s) \\
\quad \textbf{else do } \{ \\
\qquad bs \leftarrow \mathsf{uniform}\ \{0,1\}^n; \\
\qquad \mathsf{return}_{\mathsf{sprob}}\ (bs, s(x \mapsto bs))\ \}
\end{array}
$$

Random oracle with map $s : \alpha \rightharpoonup \{0,1\}^n$

Keep track of queries in Q

$\mathsf{RO}(s, x) =$
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$\mathsf{RO}'((s, Q), x) =$
if $x \in \mathrm{dom}(s)$ **then**
  $\mathrm{return}_{\mathrm{sprob}}\ (s(x), (s, Q \cup \{x\}))$
else do $\{$
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1. Define relation on states: $\overline{s\ \mathrm{S}\ (s, Q)}$
2. Prove bisimulation property:
   - $\varnothing\ \mathrm{S}\ (\varnothing, \{\})$
   - If $s\ \mathrm{S}\ sQ$, then $\mathrm{RO}(s, x) \uparrow(=) \times S \uparrow_{\mathrm{sprob}} \mathrm{RO}'(sQ, x)$.
3. Appeal to representation independence:

$b \leftarrow \mathrm{uniform}\ \{0,1\}$;
$(m_0, m_1, \sigma, s_{\mathcal{O}}) \leftarrow \mathrm{exec}(\mathcal{A}.\mathrm{gen}(pk), \mathrm{RO}, \varnothing)$;
assert $(\text{valid-plain}(m_0) \wedge \text{valid-plain}(m_1))$;
$c^* \leftarrow \mathrm{aenc}(pk, \textbf{if}\ b\ \textbf{then}\ m_0\ \textbf{else}\ m_1)$;
$(b', \_) \leftarrow \mathrm{exec}(\mathcal{A}.\mathrm{guess}(c^*, \sigma), \mathrm{RO}, s_{\mathcal{O}})$;
$\mathrm{return}_{\mathrm{spmf}}\ (b = b')$

Random oracle with map $s : \alpha \rightharpoonup \{0,1\}^n$ — Keep track of queries in Q

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Summary

- Probabilistic language with oracles in HOL

$$\mathsf{gpv} \cong (\beta + \gamma \times \mathsf{rpv})\ \mathsf{sprob}$$
$$\mathsf{rpv} \cong \rho \to \mathsf{gpv}$$

- Relational parametricity yields reasoning principles

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- Shallow embedding yields short proofs

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- Shallow embedding yields short proofs

Next steps

- Formalise a computational soundness proof in the framework
- Explore the connection between parametricity and other relational program logics