Inductive theorem proving based on tree grammars

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Introduction

Why is inductive theorem proving difficult?

- Only *restricted* cut-elimination in arithmetic
  $\Longrightarrow$ No subformula property
- Induction invariants necessarily non-analytic
  - Theoretical results: $\mathrm{Con}_{I\Sigma_k} \in \Pi_1$, but $I\Sigma_k \nvdash \mathrm{Con}_{I\Sigma_k}$
  - Practical experience: loop invariants, math. proofs, ...
- How to find **non-analytic** key lemma / generalization?

Our Approach

- Consider proofs of instances $A(n)$ of $\forall x\, A(x)$
  E.g. bounded model checking, constructive ω-rule, ...

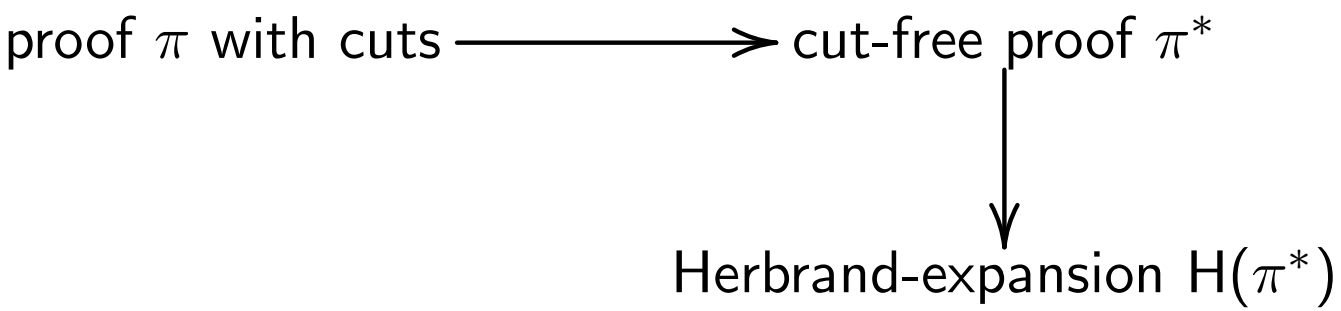
- Proof-theoretic analysis of structure of instance-proofs
  - Based on Herbrand's theorem and tree grammars
  - Extending method for cut-introduction to induction

Outline

- Cut-elimination and cut-introduction
- Inductive theorem proving based on tree grammars

Cut-elimination and Herbrand's theorem

- **Herbrand's theorem** (1930). A first-order formula φ is valid iff a finite expansion of φ is a tautology.
  E.g. $\exists x\, A_{\text{qf}}(x)$ expands to $A_{\text{qf}}(t_1) \vee \ldots \vee A_{\text{qf}}(t_n)$
- In total:



Cut-elimination and tree grammars

- Information of Herbrand-expansion $A_{\text{qf}}(t_1) \vee \ldots \vee A_{\text{qf}}(t_n)$ is $T = \{t_1, \ldots, t_n\}$ – a (finite) tree language.

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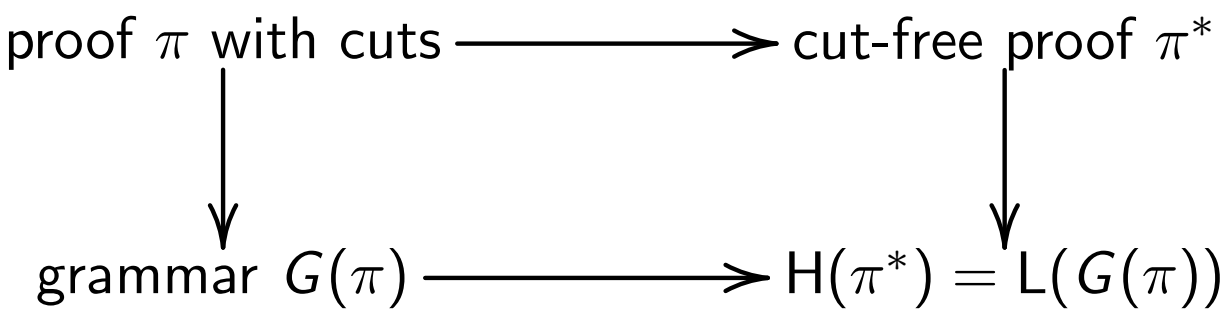
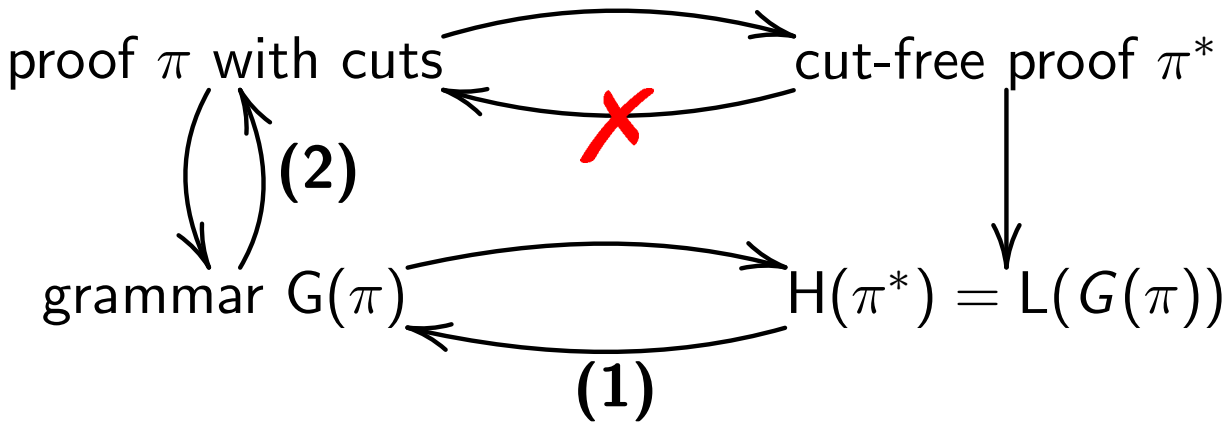
- **Theorem** (H 2012). If π is a proof with Π_1-cuts then there is a totally rigid acyclic tree grammar $G(\pi)$ s.t. $L(G(\pi))$ is a Herbrand-expansion and $|G(\pi)| \leq |\pi|$.
- In total:



Cut-introduction based on tree grammars

- Cut-introduction method [H, Leitsch, Reis, Weller '12, '14]



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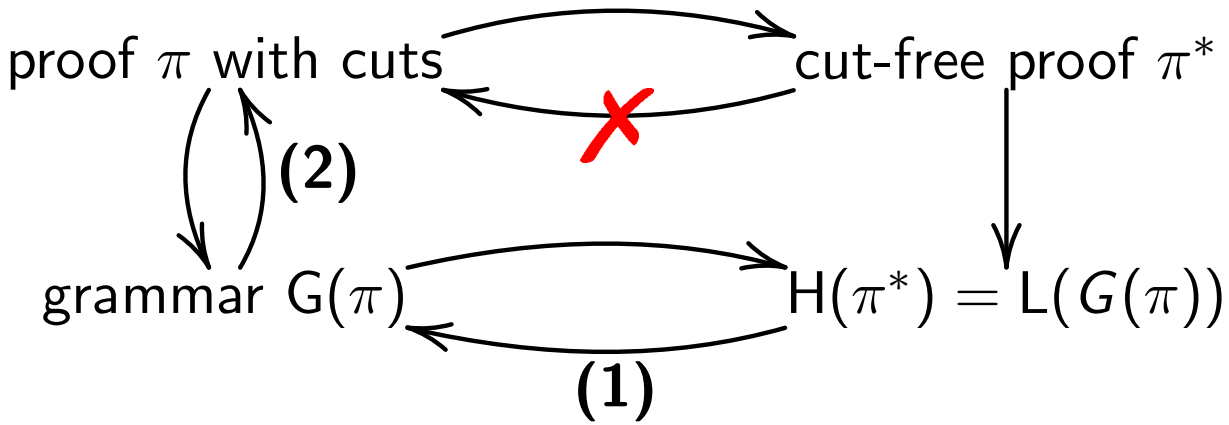

1. Given L, compute G s.t. $L(G) = L$ (or $L(G) \supseteq L$).
2. Solve Boolean unification problem with predicates, e.g.

$$S(X) \quad \equiv \quad X(\alpha) \to (X(t_1) \wedge X(t_2)) \vdash A(u_1), A(u_2), A(u_3)$$

Find F s.t. $S(F)$ is a tautology

- Quantifier-free
- Every solution induces proof with cuts
- For $S(X)$ induced by grammar for Π_1-cuts: canonical solution

Outline

- ✓ Cut-elimination and cut-introduction
- ▶ Inductive theorem proving based on tree grammars

Cut and induction

$$\dfrac{\begin{array}{cc}(\pi_b) & (\pi_s(\alpha)) \\ \Gamma \vdash A(0) & \Gamma, A(\alpha) \vdash A(s\alpha)\end{array}}{\Gamma \vdash A(t)}\ \text{ind}$$

If t is variable-free, there is $n \in \mathbb{N}$ s.t. $|t| = n$

$$\dfrac{\dfrac{\begin{array}{cc}(\pi_b) & (\pi_s(0)) \\ \Gamma \vdash A(0) & A(0), \Gamma \vdash A(s0)\end{array}}{\Gamma \vdash A(s0)}\ \text{cut}}{\begin{array}{c}\vdots \\ \Gamma \vdash A(s^n 0)\end{array} \qquad A(s^n 0) \vdash A(t)}\ \text{cut}$$

$\Rightarrow$ Induction is infinitary cut

Simple induction proofs

No nested induction, Π_1-invariant

$$\dfrac{\dfrac{\Gamma \vdash \forall y\, F(0,y) \quad \Gamma, \forall y\, F(\nu,y) \vdash \forall y\, F(s\nu,y)}{\Gamma \vdash \forall y\, F(\alpha,y)} \qquad \Gamma, \forall y\, F(\alpha,y) \vdash B(\alpha)}{\dfrac{\Gamma \vdash B(\alpha)}{\Gamma \vdash \forall x\, B(x)}}\ \text{cut}$$

Simple induction proofs: example

- Background theory Γ:

$$
\begin{aligned}
&f(0) = 1\\
&f(sx) = sx \cdot f(x)\\
&g(x,0) = x\\
&g(x,sy) = g(x \cdot sy, y)\\
&x \cdot 1 = x\\
&1 \cdot x = x\\
&(x \cdot y) \cdot z = x \cdot (y \cdot z)
\end{aligned}
$$

- Want to prove: $g(1,\alpha) = f(\alpha)$.

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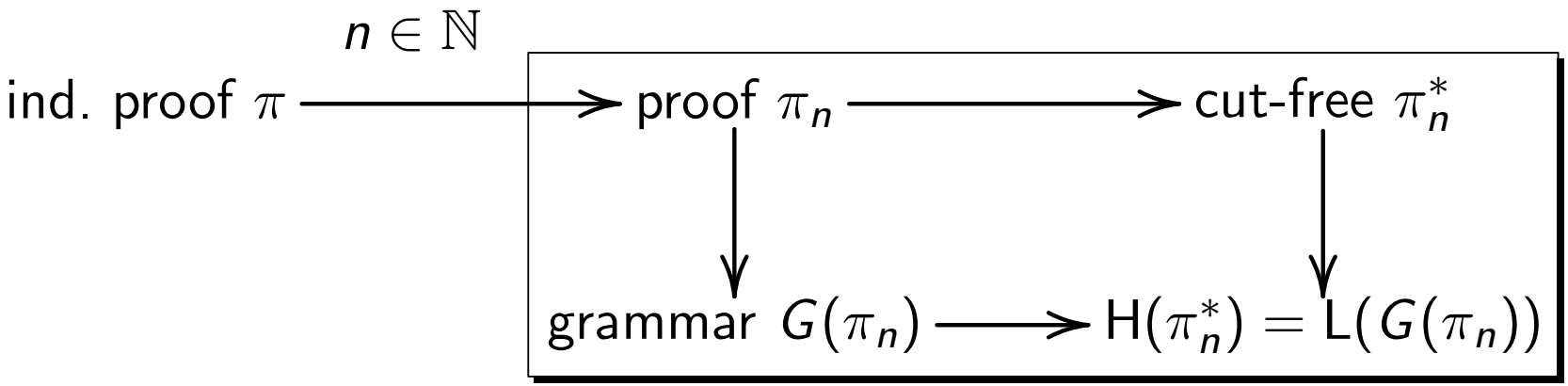
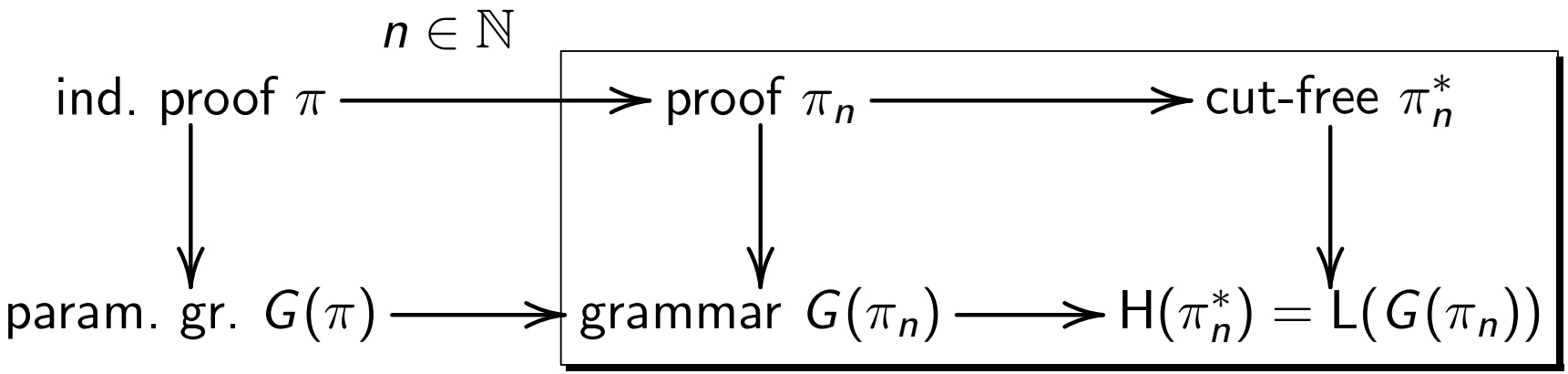
- Want to prove: $g(1,\alpha) = f(\alpha)$.
- Needs generalisation to: $\forall y\, g(y,\alpha) = y \cdot f(\alpha)$

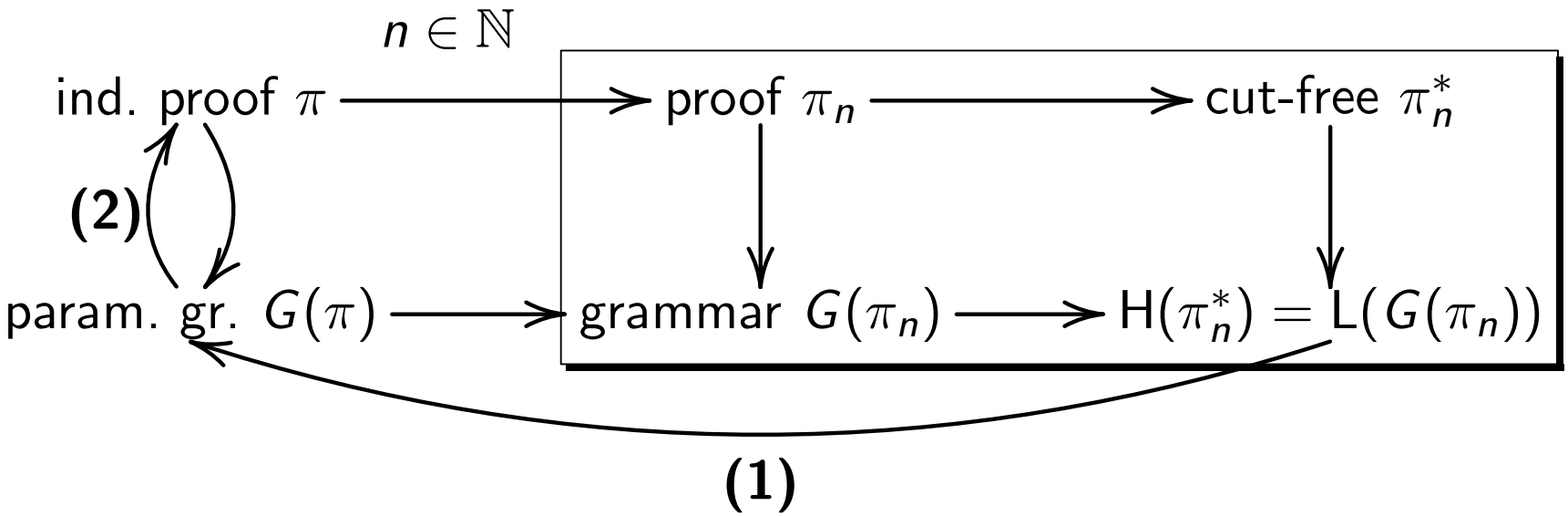
Induction and tree grammars (1/3)



Induction and tree grammars (2/3)



Induction and tree grammars (3/3)



The algorithm

Input: background theory Γ, statement $\forall x\, A(x)$

1. Compute instance proofs $\{\pi_i^* : A(i) \mid i \in M \subseteq_{\text{fin}} \mathbb{N}\}$
2. Compute parametric grammar G s.t. $L(G_i) \supseteq H(\pi_i^*)$ for $i \in M$
3. $S(X) :=$ unification problem induced by G
4. (Try to) find an F s.t. $S(F)$ is a tautology
5. $\pi :=$ proof induced by $S(F)$

Output: inductive proof π of $\Gamma \vdash \forall x\, A(x)$

Computation of grammar

- **Problem.** Given $\{H(\pi_i^*) \mid i \in M \subseteq_{\text{fin}} \mathbb{N}\}$ compute parametric grammar G s.t. $\mathrm{L}(G_i) \supseteq H(\pi_i^*)$ for all $i \in M$.
- Find a smallest such G by PTIME-translation to MaxSAT
- No coverage-guarantee for all $i \in \mathbb{N}$

Solving the unification problem

- **Problem.** Solve unification problem of the form:
  $\Gamma_0 \vdash X(0, \beta)$
  $\Gamma_1, \bigwedge_{1 \leq i \leq n} X(\nu, t_i[\alpha, \nu, \gamma]) \vdash X(s\nu, \gamma)$
  $\Gamma_2, \bigwedge_{1 \leq i \leq m} X(\alpha, u_i[\alpha]) \vdash B(\alpha)$
- Quantifier-free
- Every solution gives inductive proof
- In general: undecidable, but ...
- Complete algorithm
- Fast heuristics
- Decidable classes, e.g. parametric grammar with affine productions based on Karr's program analysis

Summary

Gist of this approach:

- Reduction of problem with quantifiers to one without
  - Compute parametric grammar based on ground terms only
  - Parametric grammar fixes many aspects of induction proof
- Rudimentary implementation finds non-analytic invariants

Towards an inductive theorem prover:

- Decidable subclasses of Boolean unification with predicates
- TIP library: many-sorted, algebraic data types
- Generalize proof shape
- Work modulo background theory

Further future work:

- Completeness
- Π_n-cuts and -induction