Verifying Auctions

Manfred Kerber  Marco Caminati
Christoph Lange  Colin Rowat

University of Birmingham
Computer Science  Economics
www.cs.bham.ac.uk/research/projects/formare/

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Overview

Motivation:

- Proofs in economics use often undergraduate level maths
- Proofs in economics are error prone (just as in any other theoretical fields)
- Formalization should be achievable – not just for computer scientists, but also for economists (?)
- Understand problems with the usage of theorem proving systems (!)
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Outline

- Related Work
- Pillage games
- Auction theory
- Open questions
- Summary
Some Related Work
Arrow’s impossibility theorem

A constitution respects **UN** if society puts alternative \( a \) strictly above \( b \) whenever every individual puts \( a \) strictly above \( b \). The constitution respects **IIA** if the social relative ranking (higher, lower, or indifferent) of two alternatives \( a \) and \( b \) depends only on their relative ranking by every individual. The constitution is a **D** by individual \( n \) if for every pair \( a \) and \( b \), society strictly prefers \( a \) to \( b \) whenever \( n \) strictly prefers \( a \) to \( b \). [Geanakoplos 05]
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**Theorem (Arrow – 3 Proofs by Geanakoplos 2005)**

(For two or more agents, and three or more alternatives,) any constitution that respects transitivity, **IIA**, and **UN** is a **D**.
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“When applied to a space of 20 principles for preference extension familiar from the literature, this method yields a total of 84 impossibility theorems, including both known and nontrivial new results.” [Geist-Endress-11]
References


References (Cont’d)


Pillage Games
Given a resource allocation $X = \{x_i \mid i \in I \land x_i \geq 0, \sum_{i \in I} x_i = 1\}$, the following axioms can be defined. A power function $\pi$ satisfies

**WC** (weak coalition monotonicity)
if $C \subseteq C' \subseteq I$ then $\pi(C, x) \leq \pi(C', x) \forall x \in X$;

**WR** (weak resource monotonicity)
if $y_i \geq x_i \forall i \in C \subseteq I$ then $\pi(C, y) \geq \pi(C, x)$; and

**SR** (strong resource monotonicity)
if $\emptyset \neq C \subseteq I$ and $y_i > x_i \forall i \in C$ then $\pi(C, y) > \pi(C, x)$. 

Wealth Is Power

\[ \text{WIP}_\pi[C, x] := \sum_{i \in C} x_i \]
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Stable Set: \[ S = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (0, \frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{1}{4}, \frac{1}{2}, \frac{1}{4}), (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})\} \]
Some Results

Formalization: Theorema I. Represent the main definitions and results

Proofs: Prove some theorems in Theorema

Pseudo Algorithm: Summarize the results in a Theorema algorithm with oracle, where the oracle is given by lemmas which can be proved in Theorema.

Presentation at ICE 2012 (Initiative for Computational Economics, ice.uchicago.edu/) ~ look into other areas.

We organized a symposium at this year’s AISB convention on Do-Form: Enabling Domain Experts to use Formalised Reasoning
www.cs.bham.ac.uk/research/projects/formare/events/aisb2013
Auctions
Auctions

Auctions allocate trillions of dollars in goods and services every year.

Given: a set of individual bids for a good (not necessarily the same value an individual ascribes to the good!)

Goals:
- give the good to the bidder who values it most
- determine prices
- maximize revenue

New auctions are designed and some properties are proved.

Strict rules must be followed.

New auctions may have problems.
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- New auctions may have problems.
**Formal Specification**  
(written by Isabelle user, reviewed by auction designer)

- **Theorems**
- **Definitions**

state properties of

High-level outline of our approach
From Specifications to Software

Formal Specification
(written by Isabelle user, reviewed by auction designer)

Theorems

Proofs
(checkered by Isabelle)

Definitions

Code
(executable Scala)

state properties of

implement

generate

prove

High-level outline of our approach
Second-price auction: *a highest bidder wins, pays highest remaining bid.*

**Theorem (Vickrey 1961)**

*In a second-price auction, “truth-telling” (i.e. submitting a bid equal to one’s actual valuation of the good) is a weakly dominant strategy. The auction is efficient.*

- earliest result in modern auction theory
- simple environment in which to gain intuitions
Weakly Dominant Strategy

A definition

Given some auction, a strategy profile $b$ supports an equilibrium in weakly dominant strategies if, for each $i \in N$ and any $\hat{b} \in \mathbb{R}^n$ with $\hat{b}_i \neq b_i$, $u_i(\hat{b}_1, \ldots, \hat{b}_{i-1}, b_i, \hat{b}_{i+1}, \ldots, \hat{b}_n) \geq u_i(\hat{b})$. I.e., whatever others do, $i$ will not be better off by deviating from the original bid $b_i$. 
VCG Mechanism to Auction a Set of Goods

- Extension of a single good auction to a simultaneous auction of a set of goods (by Vickrey, Clarke, Groves), a so-called ‘combinatorial’ auction.
- Users can bid on any (non-empty) subset.
- Second price mechanism.
- Computationally expensive
An Impression of the Type of Maths Involved

\[ X^* \in \arg \max_{X_1, \ldots, X_N} \sum_{n=1}^{N} b_n (X_n) \text{ s.t. } \bigcup_{n=1}^{N} X_n \subseteq \Omega \text{ and } X_n \cap X_{n'} = \emptyset \text{ for } n \neq n' \] (1)

at prices

\[ p_n \equiv \alpha_n - \sum_{m \neq n} b_m (X^*_m) \] (2)

where

\[ \alpha_n \equiv \max_{X_m} \left\{ \left| \sum_{m \neq n} b_m (X_m) \right| \big| \bigcup_{m \neq n} X_m \subseteq \Omega \text{ and } X_m \cap X_{m'} = \emptyset \text{ for } m \neq m' \right\} \] (3)
Experiments with 4 Systems for Single Good Auctions

1. Isabelle/HOL *(with Markarius Wenzel)*: higher-order logic (typed), interactive theorem proving environment, document-oriented IDE

2. Theorema 2.0 *(with Wolfgang Windsteiger)*: FOL + set theory, textbook-style documents (Mathematica notebooks), proof management GUI

3. Mizar *(Marco Caminati)*: FOL + set theory, text editor, proof checker

4. Hets/CASL/TPTP *(with Till Mossakowski)*: sorted FOL, text editor, proof management GUI, front-end to local or remote automated provers
Choice for Isabelle

- Strong user community
- Extended library
- In-built automatic reasoners
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- **Code extraction mechanism**: Code extraction is a major selling point, since auction designers do not doubt their theorems.
Theory graph generated by Isabelle

- [Pure]
- [HOL]
  - SetUtils
  - Code_Abstract_Nat
  - Code_Target_Nat
  - Discrete
  - Indicator_Function
  - Argmax
  - Partitions
  - RelationOperators
  - RelationProperties
  - MiscTools
  - StrictCombinatorialAuction
  - Universes
  - UniformTieBreaking
  - CombinatorialAuction
The set of maximal allocations:

abbreviation "vcgas N G b r == Outside' {seller} ' 
   ((argmax◦setsum) (randomBids' N G b r) 
   ((argmax◦setsum) b (allAllocations (N∪{seller}) (set G))))" 

The unique winning allocation after tie breaking.

abbreviation "v pca N G b r == the_elem (vcgas N G b r)"
Two possibilities to concepts such as injections:

- **Classical**: The set of all injections from a set $A$ to a set $B$ can be defined as the set of all relations $R$ with domain $A$ and a range that is a subset of $B$ such that $R$ and $R^{-1}$ are right-unique.
Two possibilities to concepts such as injections:

- **Classical:** The set of all injections from a set $A$ to a set $B$ can be defined as the set of all relations $R$ with domain $A$ and a range that is a subset of $B$ such that $R$ and $R^{-1}$ are right-unique.

- **Constructive:** Give a recursive definition (for finite sets):
  - **Base case** The injections from the empty set to an arbitrary set consists of the empty relation.
  - **Step case** Assume all injections from $A$ to $B$ given. Take an additional element $a$ not yet in $A$, extend mappings so that they map $a$ to an element that is not in the range of $A$. 
A flavour of Isabelle:

definition injections ::
  "'a set ⇒ 'b set ⇒ ('a × 'b) set set"
where "injections X Y =
  {R . Domain R = X ∧
   Range R ⊆ Y ∧
   runiq R ∧
   runiq (R⁻)}"
fun injections_alg ::
  "'a list ⇒ 'b::linorder set ⇒ ('a × 'b) set list"
where "injections_alg [] Y = [{}]
  "injections_alg (x # xs) Y =
  concat [ [ R +* {(x,y)} .
           y ← sorted_list_of_set (Y - Range R) ]
           . R ← injections_alg xs Y ]"
Why two versions and how to reconcile

- The classical definitions are closer to standard mathematical descriptions (more like in a textbook) and in consequence easier for auction designers to check.
- We need the constructive definition since only they allow to extract (Scala) code.
Bridging Theorem

Solution: Have classical plus constructive definitions and show bridging theorems.

\textbf{theorem} injections\_equiv:
\begin{verbatim}
assumes "finite Y" and "distinct X"
shows "set (injections\_alg X Y) = injections (set X) Y"
\end{verbatim}
\textbf{proof -}
\begin{verbatim}
let ?P="\lambda l. distinct l \rightarrow (set (injections\_alg l Y)=injections (set l) Y)"
have "?P []" using injections\_FromEmptyAreEmpty list.set(1) lm099 by metis
moreover have "\forall x xs. ?P xs \rightarrow ?P (x#xs)"
using assms(1) lm101 by (metis distinct.simps(2) insert_is_Un list.simps(15))
ultimately have "?P X" by (rule struct\_Induct)
then show ?thesis using assms(2) by blast
qed
\end{verbatim}

file:///home/mmk/research/formare/code/auction/isabelle/Auction/Vcg/afp/Vickrey\_Clarke\_Groves/Universes.thy (line 1221 ff)
Theorems Proved

1. VCG auctions are functions.
2. VCG allocations are pairwise disjoint.
3. In VCG allocations only goods in the auction are allocated.
4. Prices in VCG auctions are non-negative.
The Winner Determination Problem (WDP) is the same.

The prices are much easier to establish (everybody just pays their bid) so although the original proofs do not go through, Isabelle can find the proofs itself.

abbreviation "firstPriceP N \(<\text{Omega}> b r n ==
\ b (n, winningAllocationAlg N \(<\text{Omega}> r b,, n)"


The code can be extracted by an Isabelle command

\texttt{export\_code \ldots in Scala module\_name VCG file "file.scala"}

Distinguish:

- an interface (wrapper) for I/O (hand written) vs
- trusted code extracted via Isabelle for the computational part.

Example:

\texttt{file:///home/mmk/research/formare/code/auction/scala/addedWrapper.scala}
Tentative Lessons

- **Representation is non-trivial**, since it is partly not easy to understand the theorems, partly it is easy to make mistakes.
- Find mistakes by use and proof.
- Notice **hidden assumptions**
- Often proofs that look **simple, are still non-trivial** for theorem provers.
- First **rationalize proofs**.
- **HOL vs FOL, automated vs interactive ATPs** differences are not that relevant after all (but the complexity of the argument).
Further Work – Open Questions

- Extend to dynamic auctions
- Implement efficient algorithms for combinatorial auctions
- Adapt to modern auctions
- Get back to auction designers
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- Specialist knowledge is required, the systems in its current form are still difficult to use by non-experts.
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We have to work towards adjusting our methods to economics problems.

Specialist knowledge is required, the systems in its current form are still difficult to use by non-experts. Surprisingly this is no real problem. Auction designers cooperate with other experts such as lawyers. The real problem is to convince them of the usefulness of the general approach!

There are many challenging problems. Further info: www.cs.bham.ac.uk/research/projects/formare/