Distributed Markov Chains

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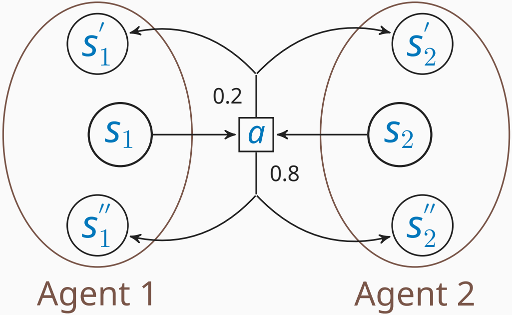
THE MODEL

Distributed Markov Chains (DMC)

- Network of communicating probabilistic transition systems
  - Synchronize on shared actions
  - Followed by joint probabilistic move
- Key restriction: no two enabled synchronizations will involve the same agent
  - Enforced syntactically (will explain soon)
- Efficient model checking using an interleaved semantics

DMC: Synchronization

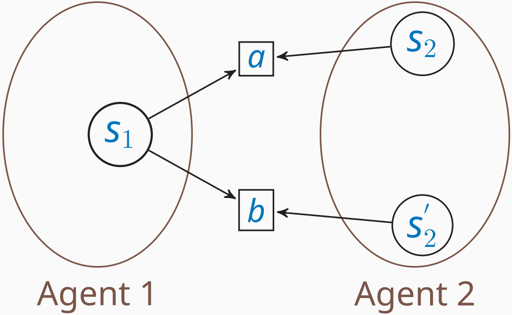
→ Joint probabilistic move after the synchronization action



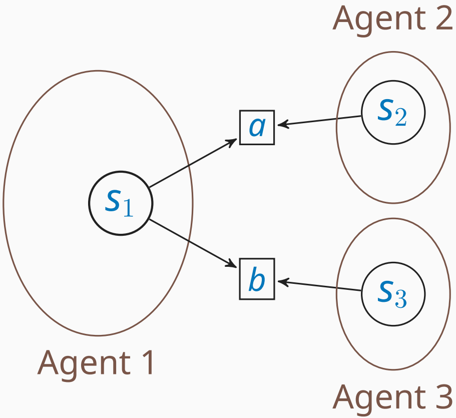
DMC: Key Restriction

- → Any two simultaneously enabled actions involve disjoint sets of agents
- → Syntactically, local state uniquely determines its communicating partners

DMC: This is allowed

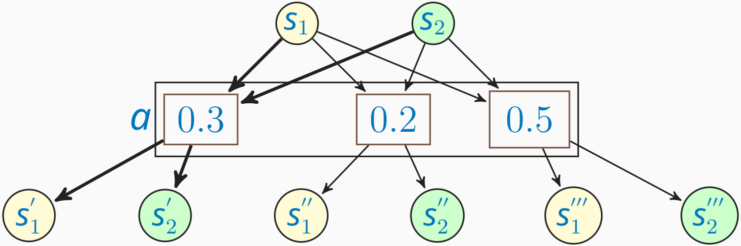


DMC: This is not allowed!



DMC: Events

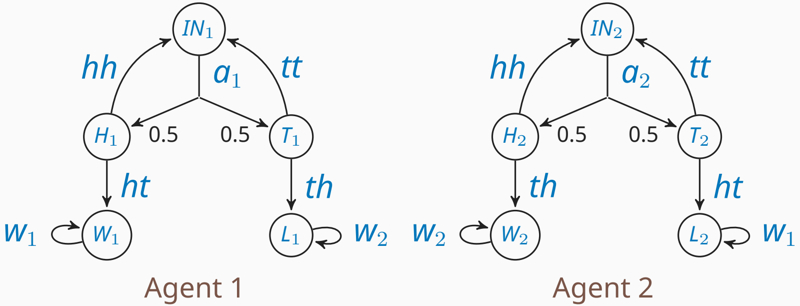
→ Event: One synchronization is executed at a time, followed by a probabilistic move by the participating agents



$e = ((s_1, s_2), a, (s_1', s_2'))$ is an event, $p_e = 0.3$

DMC: Coin Toss Example

- Two players. Each toss a fair coin
- Outcomes are the same: they toss again
- Outcomes are different: who tosses Heads wins

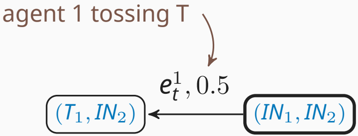


Global Transition System

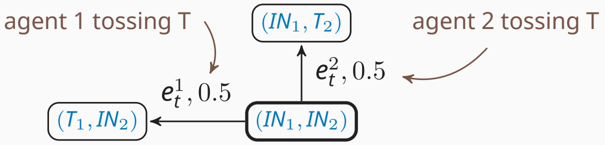
- → Associate a global transition system based on event occurrences

- → This is interleaved semantics

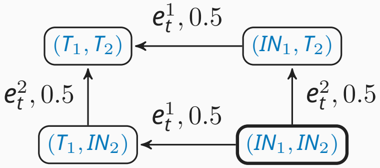
Global Transition System: Coin Toss



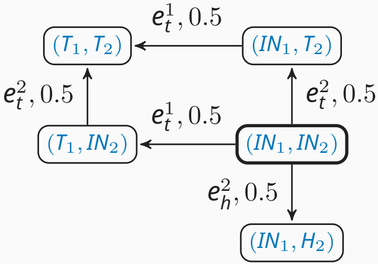
Global Transition System: Coin Toss



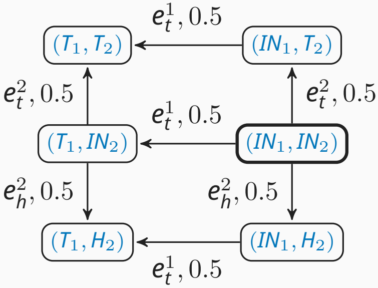
Global Transition System: Coin Toss



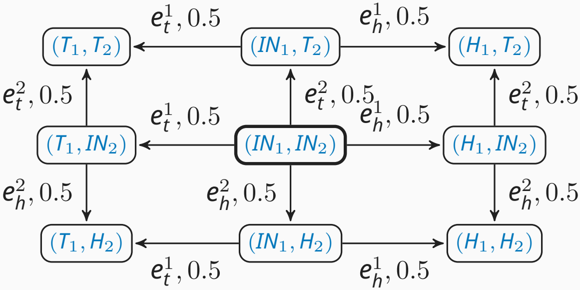
Global Transition System: Coin Toss



Global Transition System: Coin Toss

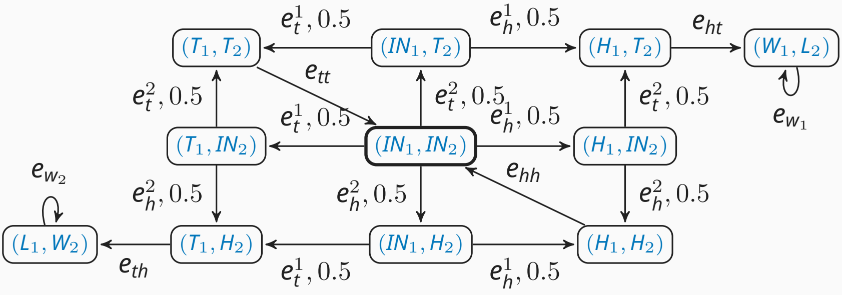


Global Transition System: Coin Toss



(both agents tossed)

Global Transition System: Coin Toss



(full global transition system)

(unmarked events have probability 1)

The Trajectory Space

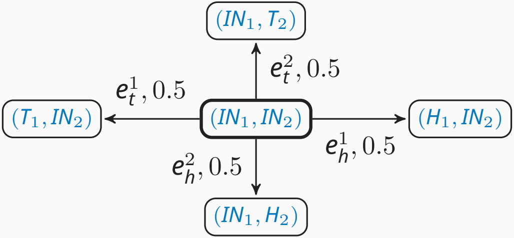
→ We refer to paths in *TS* as trajectories

→ We wish to reason about the behavior of the system using the interleaved semantics

Problem: It is *hard* to define a probability measure over the set of maximal trajectories

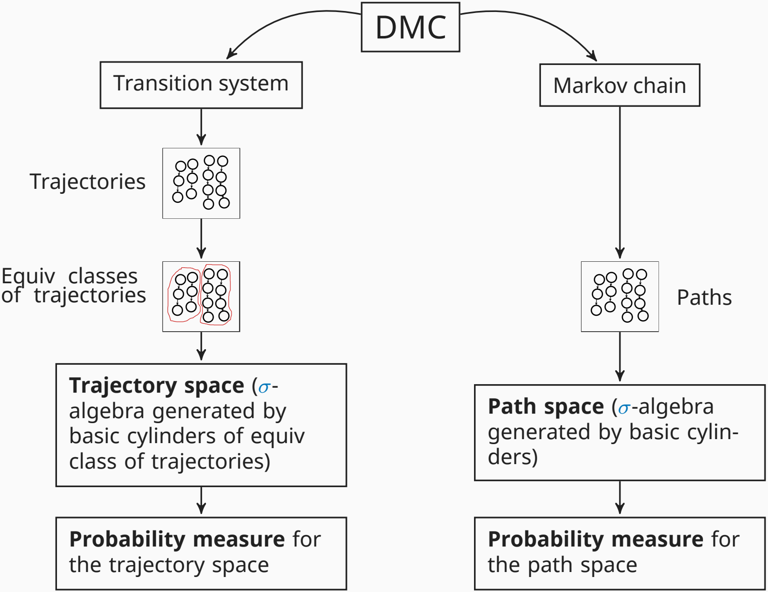
The Trajectory Space

Due to mix of concurrency and stochasticity, *TS* is not a Markov chain in general

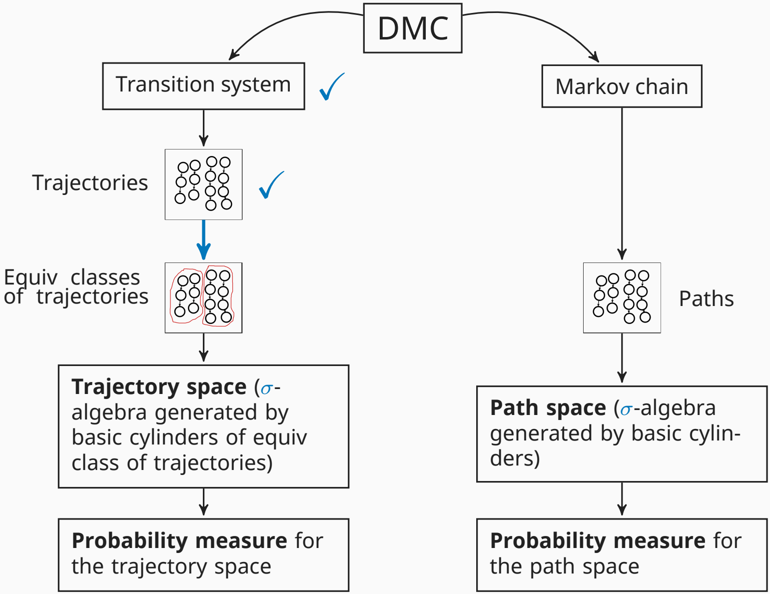


Here, the sum of the probabilities of the transitions from the state (IN_1, IN_2) is 2

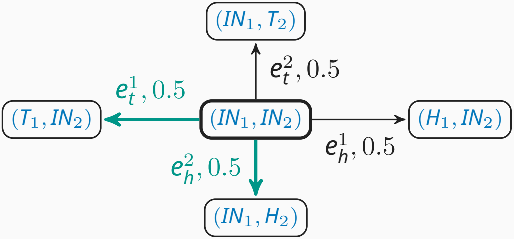
The Solution



Equivalence Classes of Trajectories

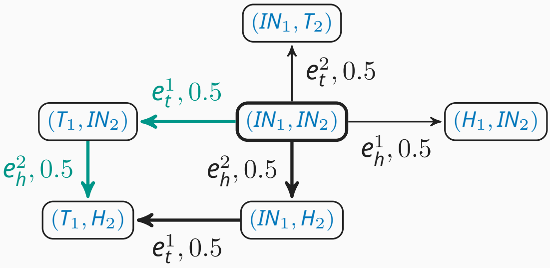


Independence over Events



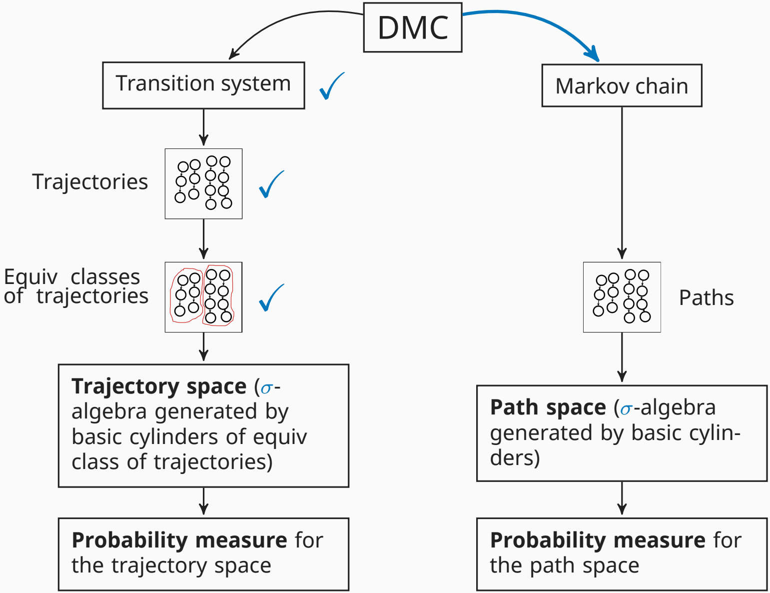
→ $e_t^1 \; I \; e_h^2$ — agent 1 tossing tail and agent 2 tossing head are independent

Equivalence over Event Sequences



- → $[e_t^1 e_h^2] = \{e_t^1 e_h^2, e_h^2 e_t^1\}$ — equivalence class over event sequences
- → Lifts to equivalence over trajectories

Markov Chain Semantics

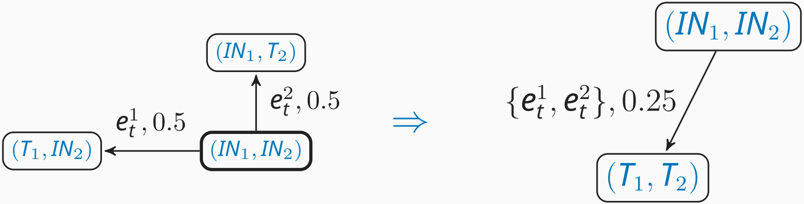


Markov Chain Semantics

→ A step at a global state is a maximal set of independent enabled events

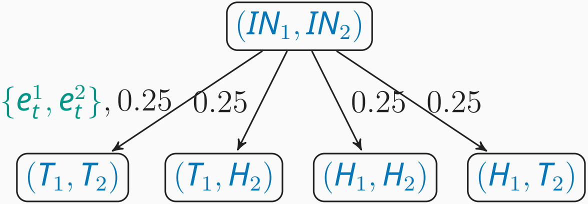
→ The transition relation using steps induces a Markov chain

Markov Chain Semantics



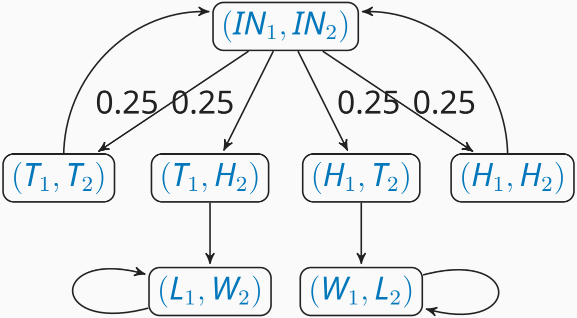
- → $\{e_t^1, e_t^2\}$ is a maximal step at (IN_1, IN_2)
- → The probability of a step is the product of probabilities associated with the events in the step

Markov Chain Semantics



The maximal steps at (IN_1, IN_2) are $\{e_h^1, e_h^2\}$, $\{e_h^1, e_t^2\}$, $\{e_t^1, e_h^2\}$, $\{e_t^1, e_t^2\}$

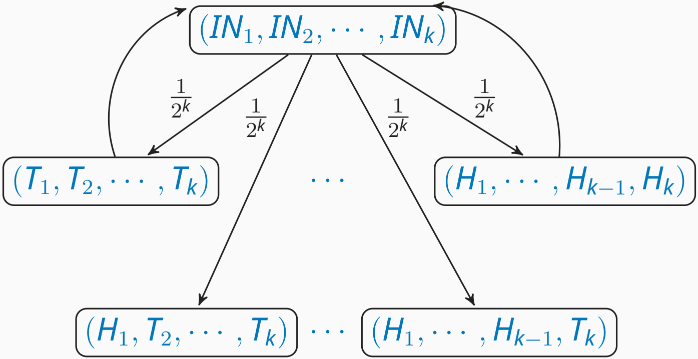
Coin Toss: Global Markov Chain



(The unmarked transitions have probability 1)

Coin Toss: Global Markov Chain

What if there were k players?



k parallel probabilistic moves generate 2^k global transitions

Markov Chain Semantics

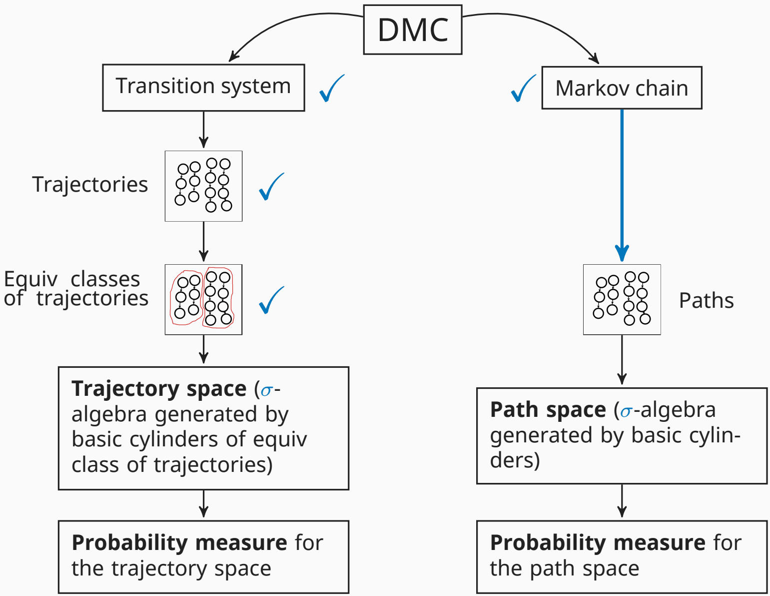
The number of transitions out of a global state can be (in number of agents)

exponential
in Markov chain semantics

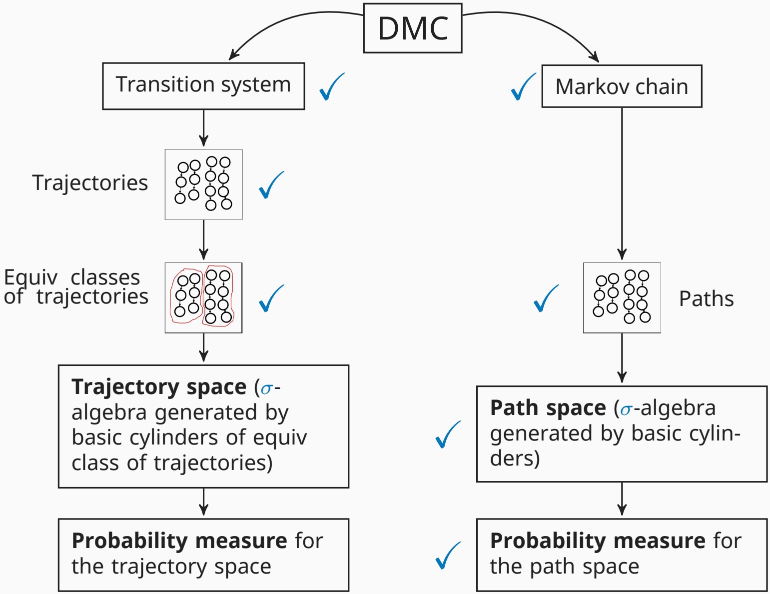
$\Rightarrow$

polynomial
in interleaved semantics

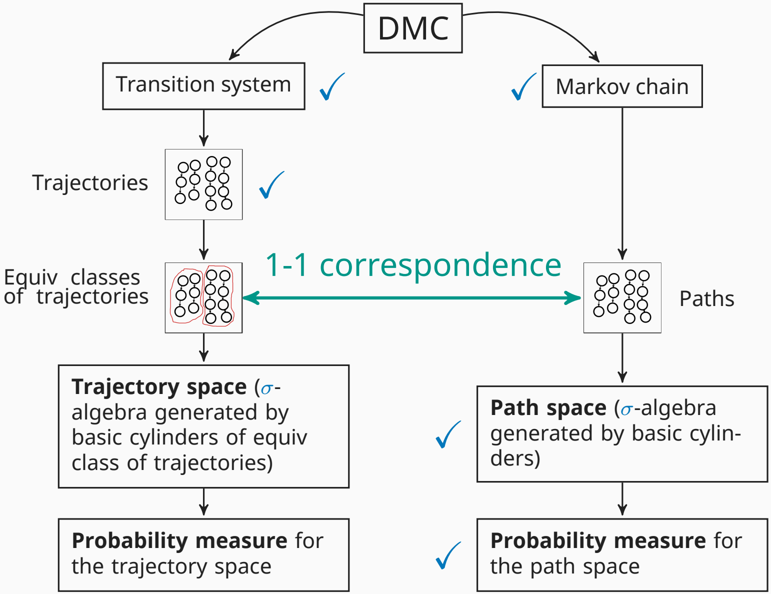
Markov Chain Semantics



Markov Chain Semantics



Defining the Probability Measure



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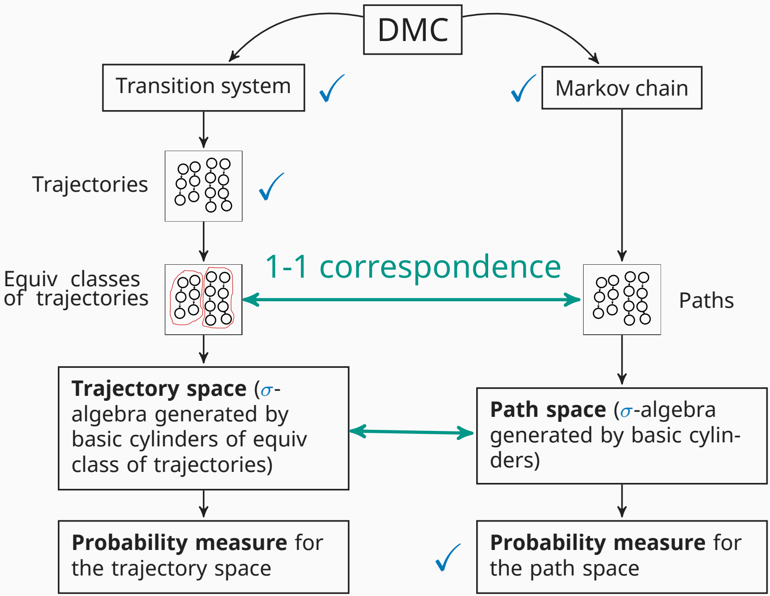
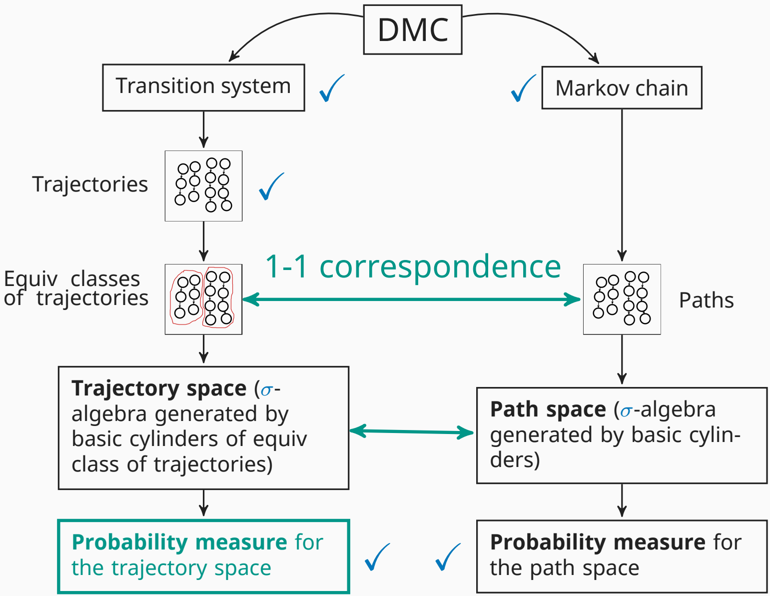


Defining the Probability Measure



THE MODEL CHECKING

$PBLTL^{\otimes}$: The Specification Logic

Local Bounded LTL ($BLTL_i$)

- Time bounded *LTL*
- Each agent i has a local set of atomic propositions AP_i
- Formula of type i: $ap \in AP_i \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \varphi_1 \mathsf{U}_i^k \varphi_2$
  - $\varphi_1 \mathsf{U}_i^k \varphi_2$ — Until holds within k (local) moves of agent i

Product Bounded LTL ($BLTL^{\otimes}$)

- Boolean combinations of $\{BLTL_i\}$ formulas

$\mathit{PBLTL}^{\otimes}$: The Specification Logic

Probabilistic Product Bounded LTL ($\mathit{PBLTL}^{\otimes}$)

→ $\mathit{Pr}_{\geq\gamma}(\varphi)$, where φ is a $\mathit{BLTL}^{\otimes}$ formula

Example (coin toss):

$$\mathit{Pr}_{\geq 0.99}\left[\left(F^7(L_1) \wedge F^7(W_2)\right) \vee \left(F^7(W_1) \wedge F^7(L_2)\right)\right]$$

with probability at least 0.99, the coin toss game terminates within 7 rounds

(the local states serve as the atomic propositions)

Statistical Model Checking

- Given a DMC and $\mathit{PBLTL}^{\otimes}$ formula $Pr_{\geq\gamma}(\varphi)$
- Explicit computation is impractical for very large systems
- Instead, estimate through sampling
  - Draw sample trajectories from the interleaved semantics
  - Use statistical estimation: returns "γ in (predefined) interval (a, b)" with high confidence

Experimental Results

→ Modeled a number of PRISM benchmarks including:

(i) Distributed leader election protocol [Itai and Rodeh]
(ii) A randomized solution to the dining Philosophers problem [Pnueli and Zuck]

→ Compared simulation time with a statistical model checker — PLASMA

Distributed Leader Election

→ Verify: with probability 1, a leader is elected eventually

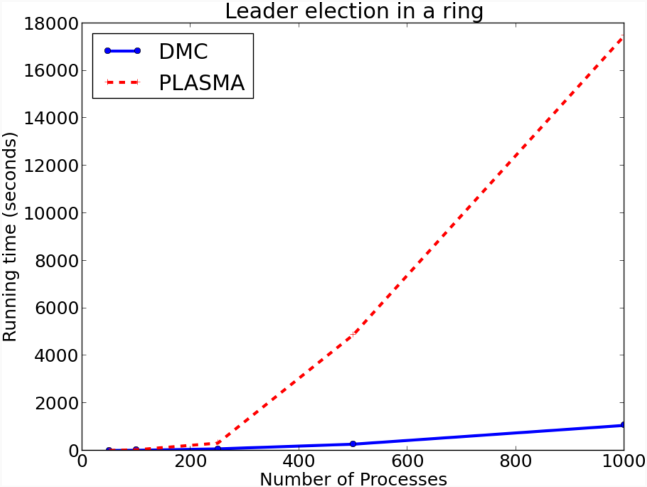
→ Since specification logic is *BLTL*$^{\otimes}$ and the verification procedure is SMC, we instead verify:

In a ring of N nodes, with high probability (p), a leader will be elected within B rounds

(Type I and II error = 0.01, indifference region = 0.01, for various choices of N and B)

Distributed Leader Election: Comparison with PLASMA

$p = 0.99$, N up to 1000 (no parallelization in DMC)



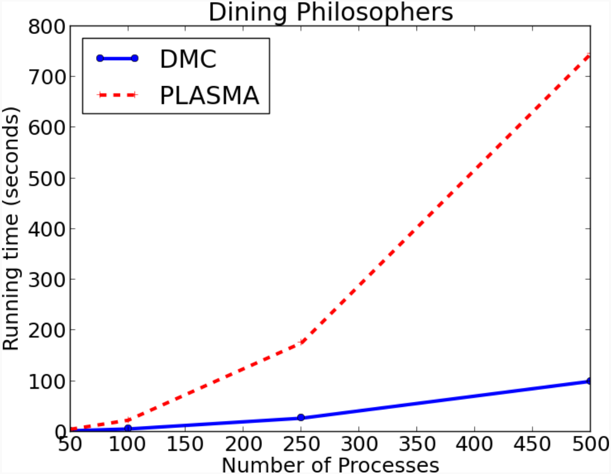
Dining Philosophers Problem

- To start, a philosopher probabilistically chooses the order in which it will try the forks
- Forks between philosophers are also agents in DMC formalism
- We use deterministic round robin protocol to simulate shared variable
- The property we verify:

  With high probability (p), every philosopher eats within B rounds

Dining Philosophers Problem: Comparison with PLASMA

$p = 0.95$, N up to 500 (no parallelization in DMC)



FUTURE WORK

Future Work and Challenges

- → Other case studies from PRISM benchmark
  - → Finding large systems with deterministic synchronizations
- → Extend SMC procedure to a parallel implementation
  - → Generating independent samples in parallel
  - → Redefining SMC parameters

Future Work and Challenges

- Use in other variations of probabilistic models
  - Currently building a probabilistic version of negotiation model
  - Reduction rules for new properties
- Model systems with observable and non-observable components
  - Model (non-)observable components as communicating agents
  - Predict behavior of non-observable agents

Future Work and Challenges

- Other application domains
  - Distributed decision making in robotics
  - Software model checking, Workflow systems
- Exact probabilistic verification
  - PCTL and other probabilistic temporal logics
  - Devising new model checking algorithms

THANK YOU!