Exercise 1

Prove (by induction over $n$) that $\frac{1}{3}n^2 + 5n + 30 \in O(n^2)$ for all $n \in \mathbb{N}^+$.

Solution:
(Note: we wouldn’t have to prove by induction, but it’s a simple case to practice it.)

$$f(n) := \frac{1}{3}n^2 + 5n + 30 \in O(n^2) \iff \exists c > 0 \exists n_0 \forall n \geq n_0 : f(n) \leq cn^2$$

Let $c := 100$, $n_0 := 1$.

Base case: $n = n_0 = 1$

$$\frac{1}{3} + 5 + 30 = 35 \frac{1}{3} \leq 100$$

Induction hypothesis: For some $n \in \mathbb{N}$: $f(n) \leq 100n^2$

Inductive step:

$$f(n + 1) = \frac{1}{3}(n + 1)^2 + 5(n + 1) + 30$$

$$= \frac{1}{3}(n^2 + 2n + 1) + 5(n + 1) + 30$$

$$= f(n) + \frac{2}{3}n + \frac{16}{3}$$

$$\leq 100n^2 + \frac{2}{3}n + \frac{16}{3}$$

$$\leq 100n^2 + 200n + 100$$

$$= 100(n + 1)^2$$

Note: we chose a pretty large $c$ for this prove – you should re-do this proof with smaller values for $c$ (such as $c = 1$) and see what happens.

Exercise 2

(a) Compare the growth of the following functions using the $o$-, $O$-, and $\Theta$-notation:

1. $n \ln n$
2. $n^l$ for all $l \in \mathbb{N}$
3. $2^n$

Hint: use L'Hôpital’s rule!

(b) Prove the following growth characterizations:

1) $\sum_{i=1}^{n} \frac{1}{i} \in \Theta(\ln n)$
2) $\ln(n!) \in \Theta(n \ln n)$

Hint: Try to prove $n^{\frac{2}{3}} \leq n! \leq n^n$ first!
Solution:

(a) \( n^l \in o(2^n) \) for all \( l \in \mathbb{N} \), because by L'Hôpital's rule:

\[
\lim_{n \to \infty} \frac{n^l}{2^n} = \lim_{n \to \infty} \frac{l \cdot n^{l-1}}{2^n \cdot \ln 2} = \lim_{n \to \infty} \frac{l \cdot (l-1) \cdot n^{l-2}}{2^n \cdot (\ln 2)^2} = \ldots = \lim_{n \to \infty} \frac{l!}{2^n \cdot (\ln 2)^l} = 0
\]

Therefore, \( n^l \in O(2^n) \) for all \( l \in \mathbb{N} \).

Obviously, \( n^1 \in o(n \ln n) \) and \( n^1 \in O(n \ln n) \), but for \( l \geq 2 \):

\[
\lim_{n \to \infty} \frac{n \ln n}{n^l} = \lim_{n \to \infty} \frac{\ln n}{n^{l-1}} = \lim_{n \to \infty} \frac{1}{n \cdot (l-1) \cdot n^{l-2}} = 0
\]

Therefore \( n^l \in \omega(n \ln n) \) for all \( l \geq 2 \). This also holds for any real \( l > 1 \).

As a consequence, \( n \ln n \in o(2^n) \).

(b) \( 1) \sum_{i=1}^{n} \frac{1}{i} \in \Theta(\ln n) \): Consider the functions \( u(x) := \frac{1}{|x|} \) and \( l(x) := \frac{1}{|x|} \), then:

\[
l(x) \leq \frac{1}{x} \leq u(x) \Rightarrow \int_{1}^{n} l(x) \, dx \leq \int_{1}^{n} \frac{1}{x} \, dx \leq \int_{1}^{n} u(x) \, dx
\]

\[
\Rightarrow \sum_{i=2}^{n} \frac{1}{i} \leq \ln n - \ln 1 \leq \sum_{i=1}^{n-1} \frac{1}{i}
\]

(draw a graph of \( u(x) \) and \( l(x) \) to see why the integrals are given by these sums).

Thus, \( \ln n \leq \sum_{i=1}^{n} \frac{1}{i} \leq \sum_{i=1}^{n} \frac{1}{i} \), and therefore \( \ln n \in O \left( \sum_{i=1}^{n} \frac{1}{i} \right) \).

As \( 2 \cdot \sum_{i=2}^{n} \frac{1}{i} = 2 \cdot \left( \frac{1}{2} + \ldots + \frac{1}{n} \right) > 1 \), we know that

\[
3 \sum_{i=2}^{n} \frac{1}{i} = 2 \sum_{i=2}^{n} \frac{1}{i} + \sum_{i=2}^{n} \frac{1}{i} > 1 + \sum_{i=2}^{n} \frac{1}{i} = \sum_{i=1}^{n-1} \frac{1}{i},
\]

and, therefore,

\[
\sum_{i=1}^{n} \frac{1}{i} < 3 \sum_{i=2}^{n} \frac{1}{i} \leq 3 \ln n \Rightarrow \sum_{i=1}^{n} \frac{1}{i} \in O(\ln n), \quad \text{q.e.d.}
\]

2) Using \( n^{\frac{n}{2}} \leq n! \leq n^n \), we get:

\[
\ln n^{\frac{n}{2}} \leq \ln(n!) \leq \ln n^n \Rightarrow \frac{n}{2} \ln n \leq \ln(n!) \leq n \ln n,
\]

which leads directly to the result \( \ln(n!) \in \Theta(n \ln n) \).

**Proof** for \( n^{\frac{n}{2}} \leq n! \leq n^n \): It is obvious that \( n! = 1 \cdot 2 \ldots n \leq n \cdot n \cdot \ldots \cdot n = n^n \). To prove \( n^{\frac{n}{2}} \leq n! \), or \( n^n \leq (n!)^2 \), we show that \( \left( \frac{n^{n}}{n^n} \right)^{\frac{n}{2}} \geq 1 \):

\[
\left( \frac{n!}{n^n} \right)^{\frac{n}{2}} = \frac{n!}{n^n} \cdot n! = \prod_{i=0}^{n-1} \frac{n-i}{n} \cdot \prod_{i=0}^{n-1} (i+1) = \prod_{i=0}^{n-1} \frac{(n-i)(i+1)}{n}
\]

and \( (n-i)(i+1) = -i^2 + ni - i + n = n + i(n-1-i) \geq n \). Therefore, all factors of the product are \( \geq 1 \). Consequently, the product itself is \( \geq 1 \).
Exercise 3

Let \( l(x) \) be the number of bits of the representation of \( x \) in the binary system. Prove:

\[
\sum_{i=1}^{n} l(i) \in \Theta(n \ln n)
\]

Solution:

We need the following equalities:

- \( \frac{n}{\ln \eta(i) = \ln \left( \prod_{i=1}^{n} i \right) = \ln(n!) \in \Theta(n \ln n) \), (see exercise 1(b), part 2!), and
- \( l(i) = \lfloor \ln_2 i \rfloor + 1 \) (if the binary representation of a number has \( l \) bits, the respective number \( i \) will be between \( 2^{l-1} \) and \( 2^l - 1 \)).

If we can show that

\[
c_1 \ln_2 i \leq \lfloor \ln_2 i \rfloor \leq \ln_2 i
\]

for some constant \( 0 < c_1 < 1 \) (the second inequality is a trivial result of the definition of \( [\cdot] \)), and use the transformation

\[
\sum_{i=1}^{n} l(i) = \sum_{i=1}^{n} (\lfloor \ln_2 i \rfloor + 1) = n + \sum_{i=1}^{n} \lfloor \ln_2 i \rfloor,
\]

we get

\[
c_1 \left( n + \sum_{i=1}^{n} \ln_2 i \right) \leq \sum_{i=1}^{n} l(i) \leq n + \sum_{i=1}^{n} \ln_2 i \Rightarrow \sum_{i=1}^{n} l(i) \in \Theta(n \ln n)
\]

We still have to prove that \( c_1 \ln_2 i \leq \lfloor \ln_2 i \rfloor \) for some \( c_1 \): For \( i \geq 3 \), we can choose \( c_1 \), such that \( i^{c_1} < \frac{i}{2} \)

(choose \( c_1 := \frac{1}{4} \), e.g.). Then

\[
c_1 \ln_2 i = \ln_2 \left( i^{c_1} \right) < \ln_2 \left( \frac{i}{2} \right) = \ln_2 i - 1 < \lfloor \ln_2 i \rfloor.
\]

As the inequality is also correct for \( i \in \{1, 2\} \), we are finished.

Exercise 4

Prove that \( \hat{\Theta} = \{(f, g) \mid f \in \Theta(g)\} \) defines an equivalence relation on the set of functions \( \{f \mid f : \mathbb{N} \rightarrow \mathbb{R} \} \).

Solution:

To show that \( \hat{\Theta} \) is an equivalence relation, we have to prove that:

- \( \hat{\Theta} \) is reflexive: as \( f \in \Theta(f) \) (e.g., choose constants \( c_1 := \frac{1}{2} \), and \( c_2 := \frac{3}{2} \)), by definition \( (f, f) \in \hat{\Theta} \);
- \( \hat{\Theta} \) is symmetric: if \( f \in \Theta(g) \), then
  \( - f \in O(g) \Rightarrow g \in \Omega(f) \)
  \( - f \in \Omega(g) \Rightarrow g \in O(f) \)
  Therefore, by definition \( g \in \Theta(f) \);
- \( \hat{\Theta} \) is transitive: if \( f \in \Theta(g) \), and \( g \in \Theta(h) \), then, there are constants \( c_1 \), \( c_2 \), \( c_3 \), and \( c_4 \), such that for sufficiently large \( n \)
  \( - c_1 f(n) \leq g(n) \leq c_2 f(n) \)
  \( - c_3 g(n) \leq h(n) \leq c_4 g(n) \)
  Therefore, \( c_1 c_3 f(n) \leq h(n) \leq c_2 c_4 h(n) \) which leads to \( f \in \Theta(h) \).
Homework

Study the following basic algorithms for sorting:

**InsertionSort:** i.e., sort a data set by successively inserting individual items into a sorted list.

**MergeSort:** i.e., splitting a list into two halves, sorting the halves individually, and merging the sorted sublists → in particular, study the Merge algorithm for combining two sorted lists into one.

You should understand how each algorithm proceeds to sort a given list of items.