Workshop in honor of Javier Esparza
ETAPS 2024

A Uniform Framework for Language Inclusion Problems

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Language Inclusion Problem

$$L \subseteq^? M$$

for L and M formal languages

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$$L \subseteq^{?} M$$

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Classical approach: complementation

$$L_1 \subseteq L_2 \iff L_1 \cap \overline{L_2} = \varnothing$$

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Classical approach: complementation

$$L_1 \subseteq L_2 \iff L_1 \cap \overline{L_2} = \emptyset$$

We present an algorithm framework avoiding explicit complementation

Algorithm Framework

$$L \subseteq^? M$$

for L and M formal languages

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The idea is to select a **subset** $S \subseteq L$ such that

1. S is a finite set $\{s_1, \ldots, s_n\}$
2. S is effectively computable
3. S contains a counter-example if one exists

Algorithm Framework

$$L \subseteq^? M$$

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$$L \subseteq M \iff S \subseteq M$$

Algorithm Framework

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for L and M formal languages

We define S via

a **fixpoint characterization** of L

+ a **quasi-order** $\preccurlyeq$ on words defined using M

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We define S via

a **fixpoint characterization** of L

+ a **quasi-order** $\preccurlyeq$ on words defined using M

We choose $S \subseteq L$ a **basis** of L for $\preccurlyeq$, i.e.

$\forall w \in L, \exists s \in S$ such that $s \preccurlyeq w$

Algorithm Framework

$$L \subseteq^{?} M$$

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We choose $S \subseteq L$ a **basis** of L for $\preccurlyeq$

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Algorithm Framework

$$L \subseteq^{?} M$$

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We choose $S \subseteq L$ a **basis** of L for $\preccurlyeq$

1. $\preccurlyeq$ is a wqo

$\Longrightarrow$

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We choose $S \subseteq L$ a **basis** of L for $\preccurlyeq$

1. $\preccurlyeq$ is a wqo $\implies$ 1. S is a finite set $\{s_1, \ldots, s_n\}$
2. $\preccurlyeq$ is monotonic $\implies$ 2. S is effectively computable

3. S contains a counter-example if one exists

Algorithm Framework

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We choose $S \subseteq L$ a **basis** of L for $\preccurlyeq$

1. $\preccurlyeq$ is a wqo	$\Longrightarrow$	1. S is a finite set $\{s_1, \ldots, s_n\}$
2. $\preccurlyeq$ is monotonic	$\Longrightarrow$	2. S is effectively computable
3. M is $\preccurlyeq$-upward closed	$\Longrightarrow$	3. S contains a counter-example if one exists

Algorithm Framework

$$L \subseteq^? M$$

for $L = L(\mathscr{G})$ for $\mathscr{G}$ a context-free grammar and M a regular language

Algorithm Framework

$$\mathcal{G}: \quad X \to aXb$$

$$X \to \varepsilon$$

$$L(\mathcal{G}) = \{a^n b^n \mid n \geq 0\}$$

Algorithm Framework

$$\mathscr{G}: \quad X \to aXb$$

$$X \to \varepsilon$$

$$F_{\mathscr{G}} : W \mapsto a\,W\,b \cup \{\varepsilon\}$$

$$L(\mathscr{G}) = \{a^n b^n \mid n \geq 0\}$$

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$\mathcal{G}: \quad X \to aXb$

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$F_{\mathcal{G}} : W \mapsto a\,W\,b \cup \{\varepsilon\}$

$F_{\mathcal{G}}(\emptyset) = \{\varepsilon\}$

$F_{\mathcal{G}}^2(\emptyset) = \{\varepsilon, ab\}$

$F_{\mathcal{G}}^3(\emptyset) = \{\varepsilon, ab, aabb\}$

...

lfp $F_{\mathcal{G}} = L(\mathcal{G})$

Algorithm Framework

Algorithm: // decide $L(\mathscr{G}) \subseteq M$
Compute $F_{\mathscr{G}}^{m}(\varnothing)$ until
$F_{\mathscr{G}}^{m-1}(\varnothing)$ is a basis of $F_{\mathscr{G}}^{m}(\varnothing)$ for $\preccurlyeq$.
Check $F_{\mathscr{G}}^{m-1}(\varnothing) \subseteq M$.

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Algorithm: // decide $L(\mathscr{G}) \subseteq M$
Compute $F^m_{\mathscr{G}}(\emptyset)$ until
$F^{m-1}_{\mathscr{G}}(\emptyset)$ is a basis of $F^m_{\mathscr{G}}(\emptyset)$ for $\preccurlyeq$.
Check $F^{m-1}_{\mathscr{G}}(\emptyset) \subseteq M$.

There is such an m because $\preccurlyeq$ is a wqo

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There is such an m because $\preccurlyeq$ is a wqo

$\preccurlyeq$ is **monotonic** if

$$u \preccurlyeq v \implies \forall w, w' \in \Sigma^* \; wuw' \preccurlyeq wvw'$$

$$F_{\mathcal{G}} : W \mapsto a\,W\,b \cup \{\varepsilon\}$$

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lfp $F_{\mathcal{G}} = L(\mathcal{G})$

By monotonicity of $\preccurlyeq$, $F_{\mathcal{G}}^{m-1}(\emptyset)$ is a basis of lfp $F_{\mathcal{G}}$ for $\preccurlyeq$

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Algorithm: // decide $L(\mathscr{G}) \subseteq M$
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There is such an m because $\preccurlyeq$ is a wqo

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By monotonicity of $\preccurlyeq$, $F_{\mathscr{G}}^{m-1}(\varnothing)$ is a basis of lfp $F_{\mathscr{G}}$ for $\preccurlyeq$

Algorithm Instantiated

Let $u \in \Sigma^*$, and $\mathscr{A}$ an NFA

$$context_{\mathscr{A}}(u) = \{\ (q_1, q_2) \mid q_1 \xrightarrow{u}_{\mathscr{A}} q_2\ \}$$

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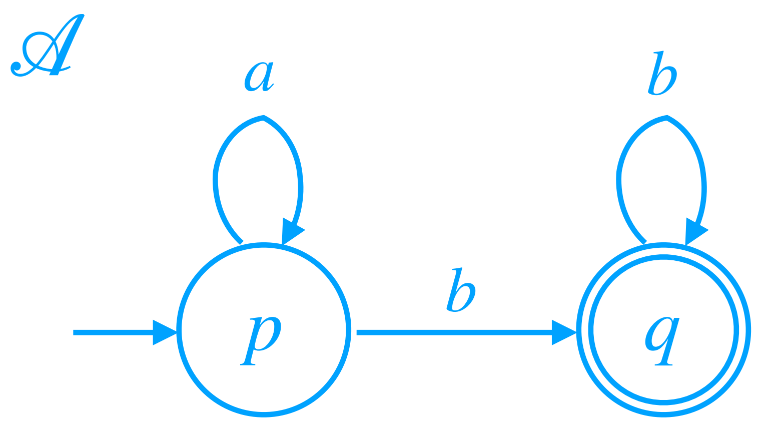

$L(\mathscr{A}) = a^*b^+$

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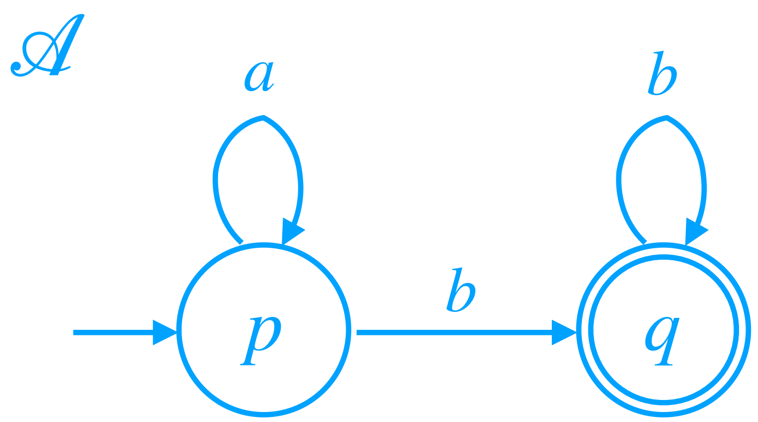

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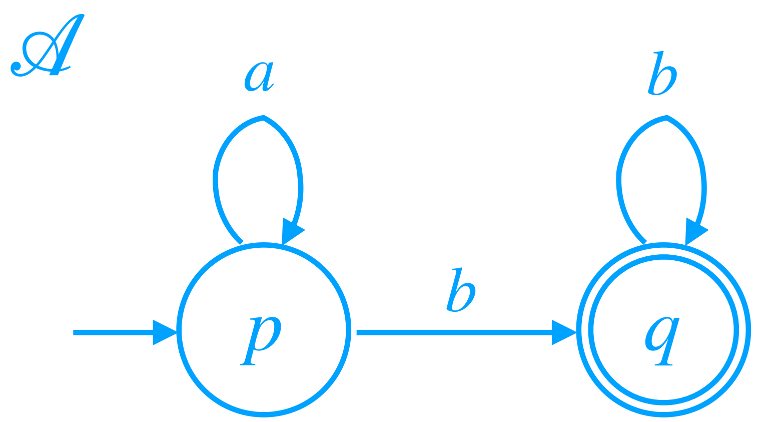

$$context_{\mathscr{A}}(\varepsilon) = \{\ (p,p),\ (q,q)\ \}$$

$context_{\mathscr{A}}(w) = \{\ (p,q)\ \}$, for all $w \in a^+b^+$

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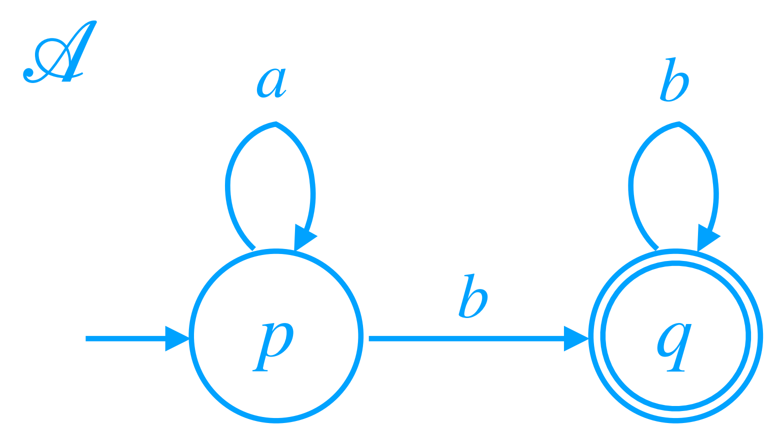

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3. $L(\mathscr{A})$ is $\preccurlyeq$-upward closed

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$$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$$

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$$F_{\mathscr{G}} : W \mapsto a\,W\,b \cup \{\varepsilon\}$$

$$F_{\mathscr{G}}(\varnothing) = \{\varepsilon\}$$

basis for $\preccurlyeq$?

$$F^2_{\mathscr{G}}(\varnothing) = \{\varepsilon, ab\}$$

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lfp $F_{\mathscr{G}} = L(\mathscr{G})$

$$\{\varepsilon, ab\} \subseteq a^* b^+$$

Algorithm Instantiated

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lfp $F_{\mathscr{G}} = L(\mathscr{G})$

$\{\varepsilon, ab\} \subseteq a^* b^+$ ❌

Algorithm Instantiated

$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$ ❌

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lfp $F_{\mathscr{G}} = L(\mathscr{G})$

$\{\varepsilon, ab\} \subseteq a^* b^+$ ❌

Algorithm Instantiated

Let $u \in \Sigma^*$, and L a regular language

$$envt_L(u) = \{ \ (w_1, w_2) \mid w_1 u w_2 \in L \ \}$$

$$u \ll v \iff envt_L(u) \subseteq envt_L(v)$$

1. $\ll$ is a wqo
2. $\ll$ is monotonic
3. L is $\ll$-upward closed

Algorithm Instantiated

Let $u \in \Sigma^*$, and L a regular language

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2. $\ll$ is monotonic
3. L is $\ll$-upward closed

$\ll$ is the coarsest order such that this is true for the case CFL into REG

State-Based Algorithm

$$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$$

$$F_{\mathscr{G}} : W \mapsto a\, W\, b \cup \{\varepsilon\}$$

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$context_{\mathscr{A}}(\varepsilon) = \{\ (p,p),\ (q,q)\ \}$

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$context_{\mathscr{A}}(\varepsilon) \subseteq context_{\mathscr{A}}(\varepsilon)$

$context_{\mathscr{A}}(ab) \nsubseteq context_{\mathscr{A}}(\varepsilon)$

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$context_{\mathscr{A}}(aabb) \subseteq context_{\mathscr{A}}(ab)$

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State-Based Algorithm

$$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$$

→ store the contexts

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lfp $F_{\mathscr{G}} = L(\mathscr{G})$

$$context_{\mathscr{A}}(ww') = \{(p,q) \mid$$
$$\exists p' .\ (p,p') \in context(w) \wedge (p',q) \in context(w')\}$$

State-Based Algorithm

$$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$$

- store the contexts
- update contexts using contexts

$$context_{\mathscr{A}}(ww') = \{(p,q) \mid$$

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lfp $F_{\mathscr{G}} = L(\mathscr{G})$

$$F^2_{\mathscr{G}}(\varnothing) = \{\varepsilon, ab\} \subseteq^? a^* b^+$$

State-Based Algorithm

$$\{a^n b^n \mid n \geq 0\} \subseteq^? a^* b^+$$

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We obtain an **antichain** algorithm

e.g. [De Wulf, Doyen, Henzinger, Raskin, CAV'06]

e.g. [Holík, Meyer, NETYS'15]

Saturation Algorithm

[Esparza, Rossmanith, Schwoon: *A Uniform Framework for Problems on Context-Free Grammars.* EATCS Bulletin, 2000] has the following algorithm

$\mathcal{G}$

$$X \to aXb$$

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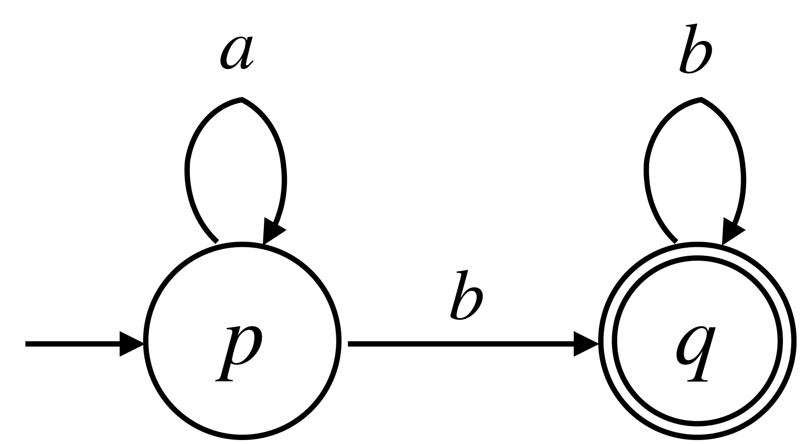

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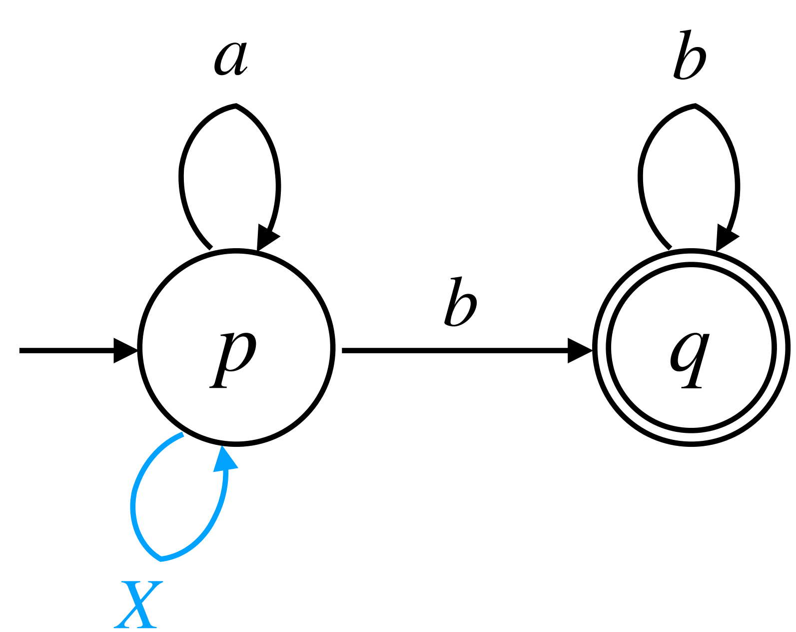

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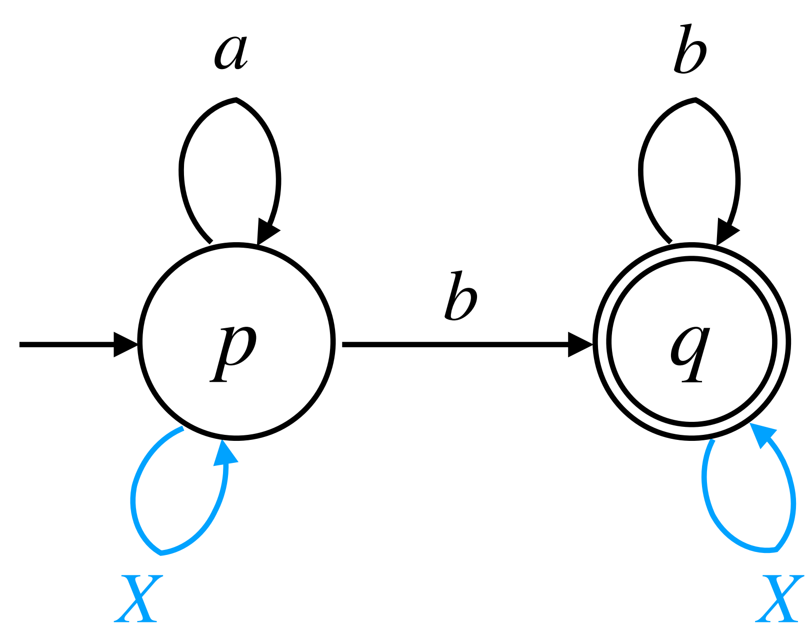

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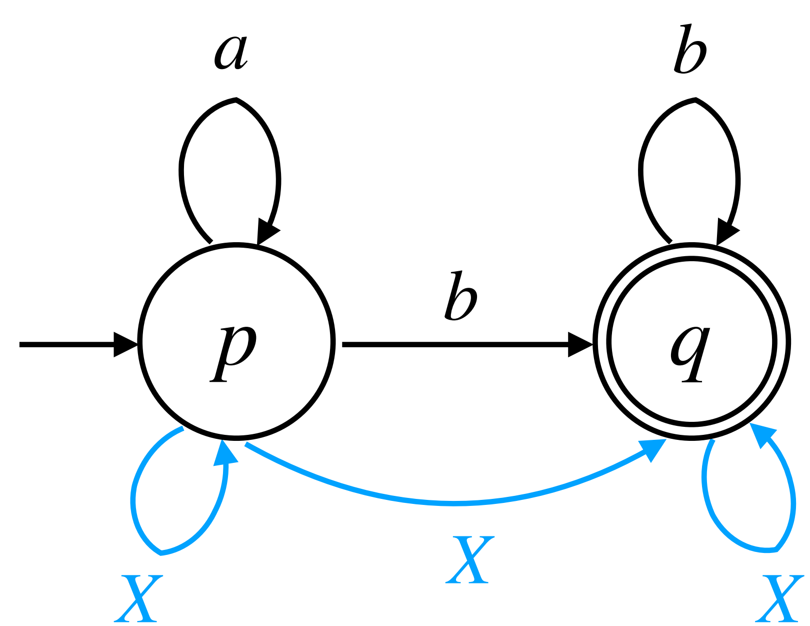

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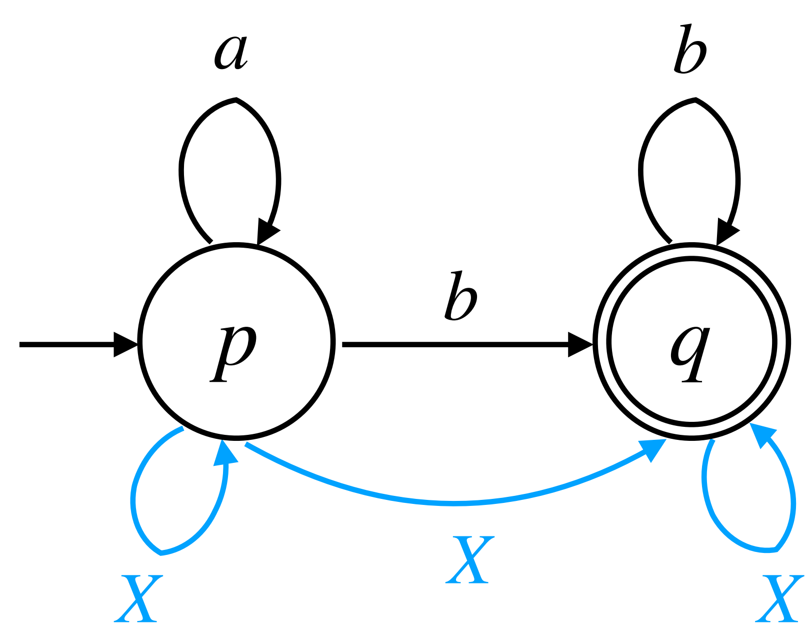

Applications to emptiness, finiteness, …

SLP Case

An SLP is a grammar generating exactly one string

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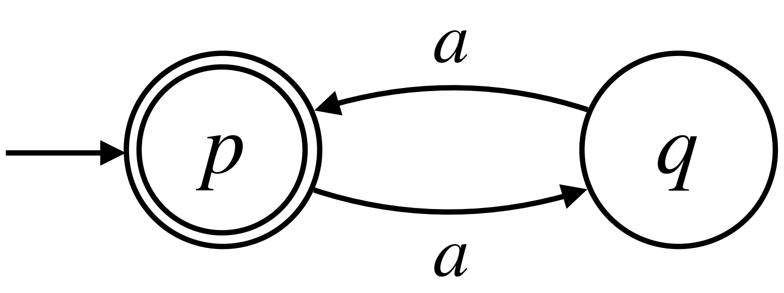

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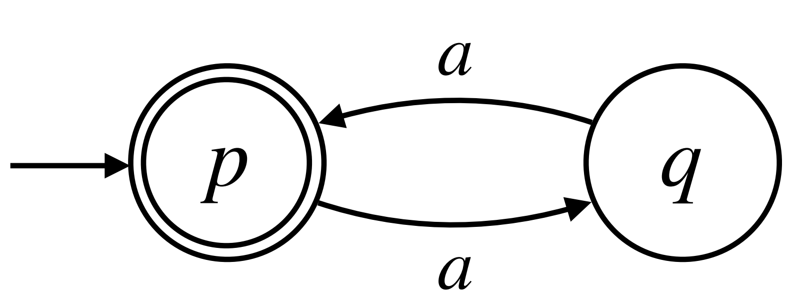

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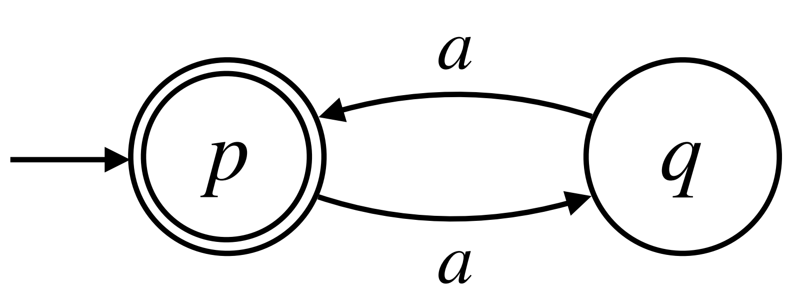

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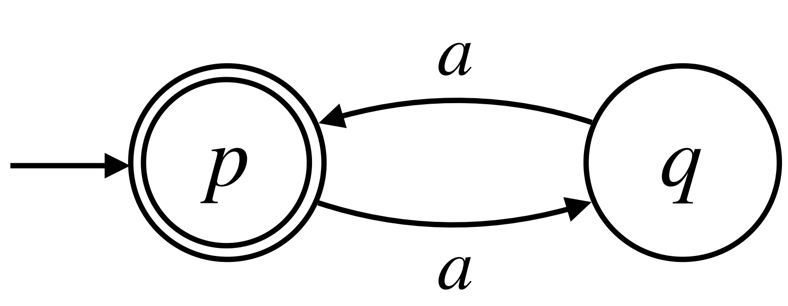

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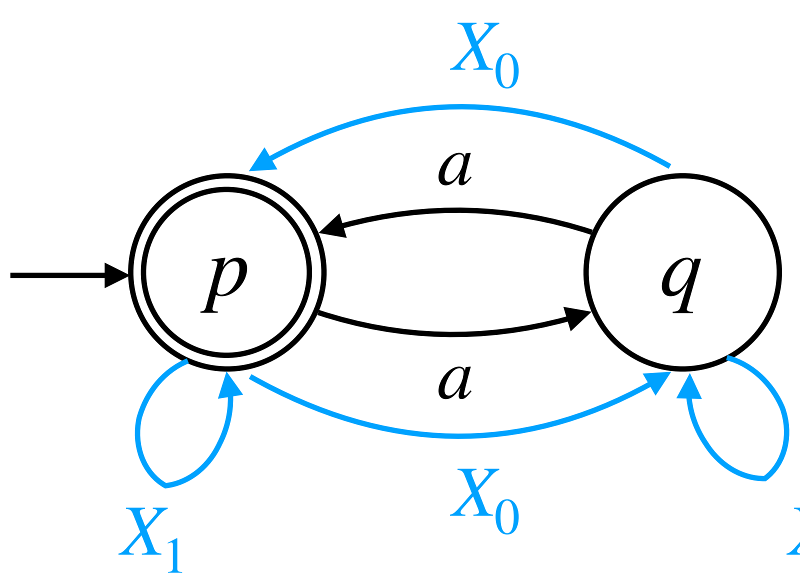

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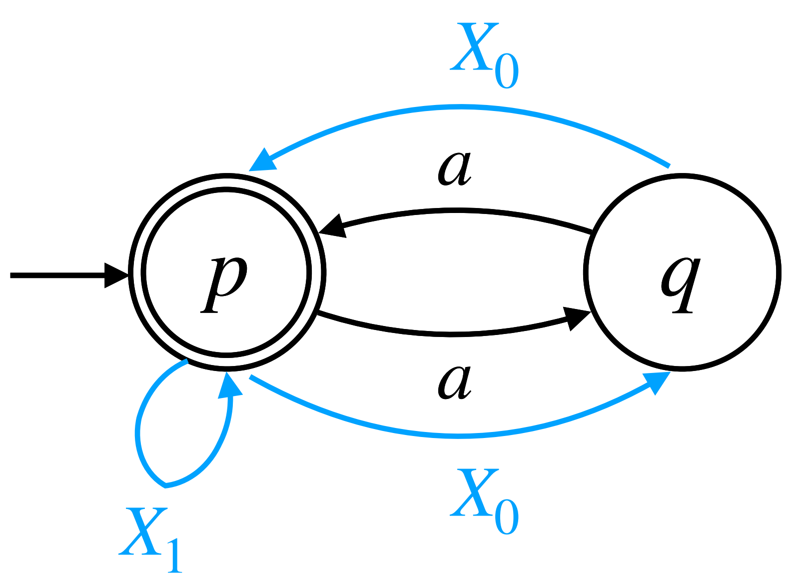

Final check: is there a X_1-arc from q_i to F?

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The case SLP $\subseteq$ REG has applications to regular expression matching for text compressed using SLP.

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$$\ldots$$
$$X_0 \to a$$

$$L(\mathcal{G}_n) = \{a^{2^n}\} \subseteq^? (aa)^*$$

Other Applications of the Framework

- REG $\subseteq$ REG
- REG $\subseteq$ Petri net traces
- ω-REG $\subseteq$ ω-REG, now part of SPOT!
- ω-CFL $\subseteq$ ω-REG
- ω-VPL $\subseteq$ ω-VPL
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Thank you for listening!

Questions?