# Model Checking Stochastic Branching Processes

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# Topic of the Talk

Two classical model-checking problems:

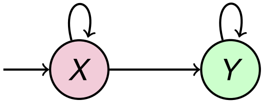
1. Does a given non-deterministic transition system satisfy a given property?
2. What's the probability that a given Markov chain satisfies a given property?

We consider linear-time properties:
$\omega$-regular specifications, e.g., LTL formulae.

Our plan:

- Define a natural generalisation of those problems.
- Solve the generalised problem.

# Nondeterministic Transition Systems

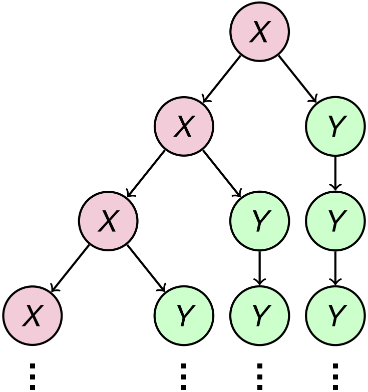



textual representation:

$$X \hookrightarrow XY \qquad Y \hookrightarrow Y$$

(one rule for each state)

induces a unique tree:



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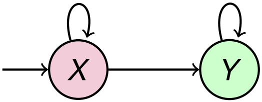


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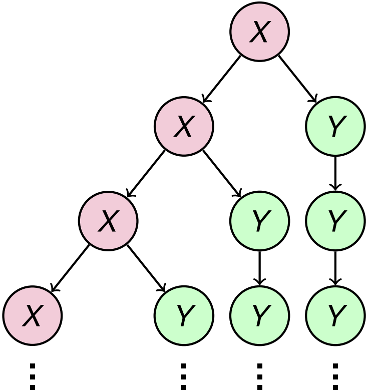


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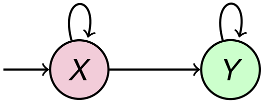


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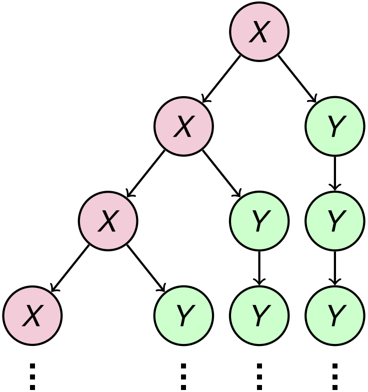


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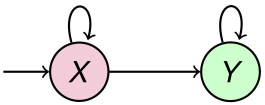


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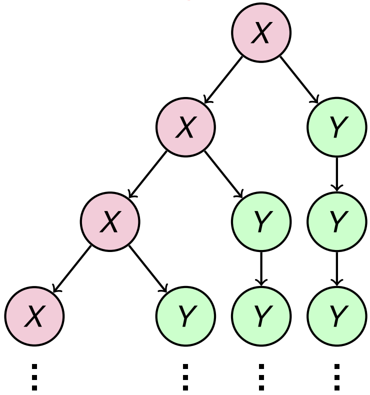


- Do all branches of the tree satisfy $\Box(Y \rightarrow \Box Y)$? Yes.
- Do all branches of the tree satisfy $\Diamond Y$?

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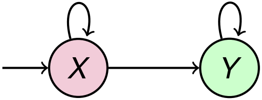


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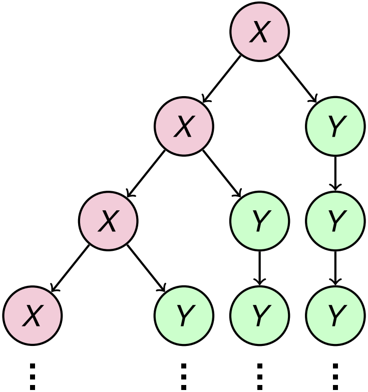


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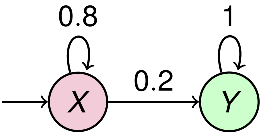
# Markov Chains

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$$X \overset{0.8}{\hookrightarrow} X \qquad Y \overset{1}{\hookrightarrow} Y$$

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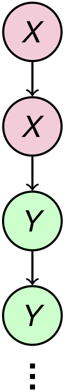
(multiple rules for each state)

induces a random "tree"
(only one branch):



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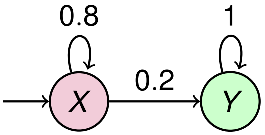


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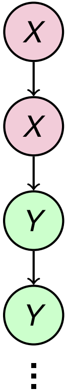


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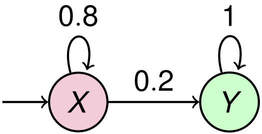


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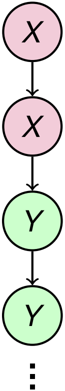


- Does the branch satisfy $\varphi_1 := \Box(Y \to \Box Y)$? $\Pr(\varphi_1) = 1$

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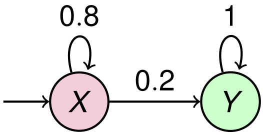


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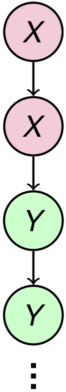


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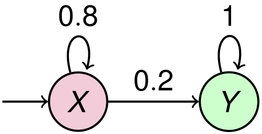
- Does the branch satisfy $\varphi_2 := \Diamond Y$?

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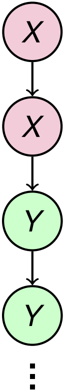


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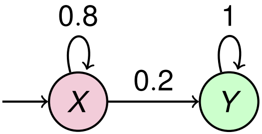


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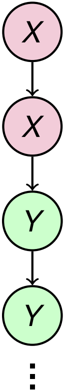


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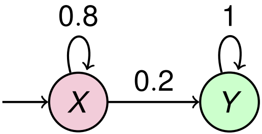
# Markov Chains

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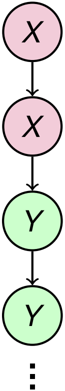


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- Does the branch satisfy $\varphi_3 := \bigcirc Y$? $\Pr(\varphi_3) = 0.2$

# Branching Processes

nondeterministic transition system:

degenerated probability distribution on trees
(probability 1 for one tree, probability 0 for all others)

Markov chain:

probability distribution on degenerated trees
(every node has just one child)

branching process:

probability distribution on trees

$$X \overset{0.6}{\hookrightarrow} X \qquad Y \overset{0.2}{\hookrightarrow} Z \qquad Z \overset{1}{\hookrightarrow} Z$$
$$X \overset{0.4}{\hookrightarrow} XY \qquad Y \overset{0.5}{\hookrightarrow} Y$$
$$Y \overset{0.3}{\hookrightarrow} YY$$

$\longrightarrow$ New plan: model check random (infinite) trees!

# Branching Processes

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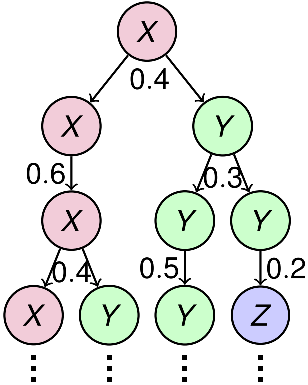


Probability of a tree that starts as on the right
$= 0.4 \cdot 0.6 \cdot 0.3 \cdot 0.4 \cdot 0.5 \cdot 0.2$

$\longrightarrow$ probability measure on (infinite) trees

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$$X \overset{0.4}{\hookrightarrow} XY \qquad Y \overset{0.5}{\hookrightarrow} Y$$
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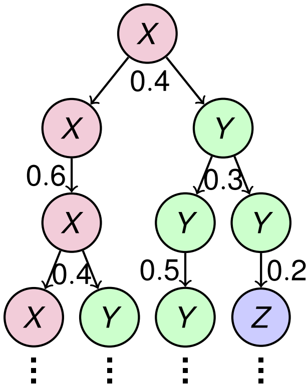
Consider $\varphi := \Box\Diamond(X \vee Z)$ (on all branches).
What is $\Pr_X(\varphi)$ ?

- "Markov-Chain" approach: $\Pr_X(\varphi) = 1$
- "Pushdown-System" approach: $\Pr_X(\varphi) = 1$



# Model Checking: Simple Doesn't Work

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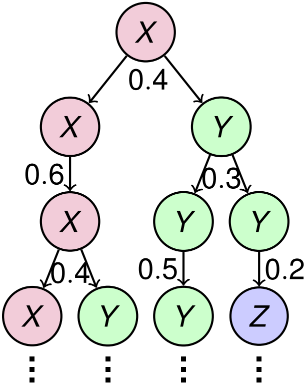


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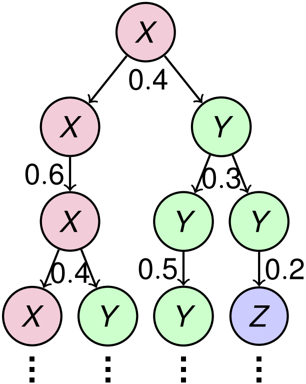


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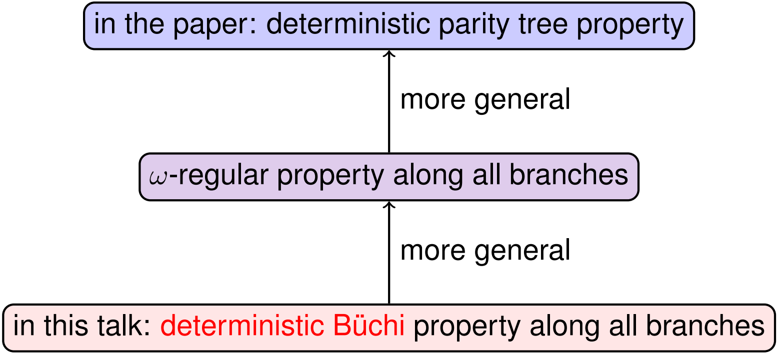
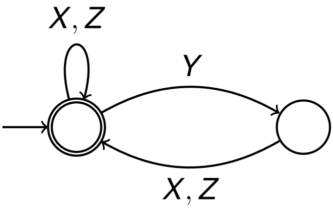
- correct value: $\Pr_X(\varphi) = 0$

However: Swapping 0.2 and 0.3 in the rules $\longrightarrow$ $\Pr_X(\varphi) = 1$.
$\Rightarrow$The exact numbers matter, even for qualitative behaviour.

# Properties


in the paper: deterministic parity tree property
more general
$\omega$-regular property along all branches
more general
in this talk: deterministic Büchi property along all branches

# Deterministic Büchi Automata




- Perform a product construction with the branching process.
  (instance of automata-theoretic approach)
- Obtain a branching process with
  accepting states and non-accepting states.

# Branching Process with (Non-)Accepting States

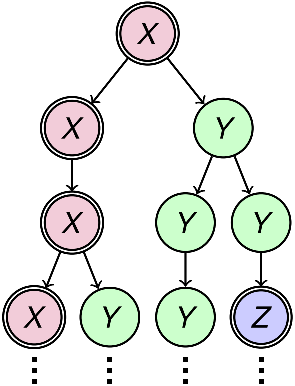
After product construction:

$X \overset{0.6}{\hookrightarrow} X$ $\quad$ $Y \overset{0.2}{\hookrightarrow} Z$ $\quad$ $Z \overset{1}{\hookrightarrow} Z$

$X \overset{0.4}{\hookrightarrow} XY$ $\quad$ $Y \overset{0.5}{\hookrightarrow} Y$

$Y \overset{0.3}{\hookrightarrow} YY$

Accepting: $X, Z$
Non-Accepting: $Y$




a tree is good $\overset{\text{def}}{\Longleftrightarrow}$
each branch has infinitely many accepting nodes

Compute $\Pr_X(\text{good})$

# Decent Trees

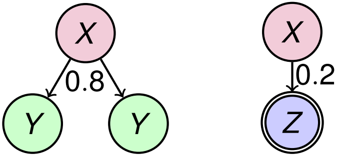
a tree is good $\overset{\text{def}}{\iff}$
each branch has $\infty$ accepting nodes

a tree is decent $\overset{\text{def}}{\iff}$
each branch has $\geq 1$ accepting node (besides the root)

| | | |
|---|---|---|
| $X \overset{0.8}{\hookrightarrow} YY$ | $Y \overset{0.3}{\hookrightarrow} XX$ | $Z \overset{1}{\hookrightarrow} X$ |
| $X \overset{0.2}{\hookrightarrow} Z$ | $Y \overset{0.7}{\hookrightarrow} Z$ | |
| $X$ non-acc. | $Y$ non-acc. | $Z$ acc. |

Equation system for $\Pr_X(\text{decent})$, $\Pr_Y(\text{decent})$, $\Pr_Z(\text{decent})$:

$$x = 0.8y^2 + 0.2$$
$$y = 0.3x^2 + 0.7$$
$$z = x$$



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$X$ non-acc. $\quad$ $Y$ non-acc. $\quad$ $Z$ acc.

Equation system for $\Pr_X$(decent), $\Pr_Y$(decent), $\Pr_Z$(decent):

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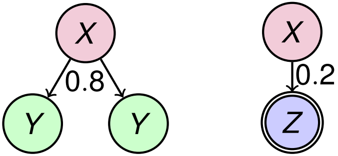


The least solution gives the correct probabilities.

# Solving Polynomial Fixed-Point Equations

$$x = 0.8y^2 + 0.2$$
$$y = 0.3x^2 + 0.7$$

One can "efficiently compute" the least solution:

**Theorem (from literature, especially [ESY, STOC'12])**

*Let $\vec{x} = \vec{f}(\vec{x})$ be an equation system where*

- *$\vec{x}$ is a vector of variables*
- *$\vec{f}$ is a vector of polynomials with nonnegative coefficients*
- *$\vec{f}(\vec{1}) = \vec{1}$ ($\vec{1}$ = vector of 1s)*

*Let q be the first entry of the least solution. Then:*

(a) *One can decide in polynomial time: Is $q = 0$? Is $q = 1$?*

(b) *One can decide in polynomial space: Is $q \bowtie \tau$? ($\tau \in \mathbb{Q}$ and $\bowtie \in \{<, >, \leq, \geq, =, \neq\}$).*

(c) *One can approximate q within additive error $2^{-j}$ in time polynomial in j and the representation size of $\vec{f}$.*

# $\Pr_X$(decent) is Easy

**Corollary**

*One can in polynomial time:*

(a) *decide whether* $\Pr_X(\textit{decent}) = 1$

(b) *"efficiently approximate"* $\Pr_X(\textit{decent})$.

Proof. Set up the equation system and apply the theorem. □

# $\Pr_X(\text{decent})$ is Easy

**Corollary**

*One can in polynomial time:*

(a) *decide whether* $\Pr_X(\textit{decent}) = 1$
(b) *"efficiently approximate"* $\Pr_X(\textit{decent})$.

Proof. Set up the equation system and apply the theorem. □

We want to do the same for $\Pr_X(\text{good})$.
First focus on the qualitative problem: Is $\Pr_X(\text{good}) = 1$ ?

# The Qualitative Problem

Generalise the notion of "decent":

a tree is *k*-decent $\stackrel{\text{def}}{\Longleftrightarrow}$
each branch has $\geq k$ accepting nodes (besides the root)

Note: 1-decent $\equiv$ decent and $\infty$-decent $\equiv$ good

a state $X$ is *k*-safe $\stackrel{\text{def}}{\Longleftrightarrow}$ $\Pr_X(k\text{-decent}) = 1$

One can compute the 1-safe states:

- remember: $X$ is 1-safe means $\Pr_X(\text{decent}) = 1$
- apply the corollary

One can compute the 2-safe states:

- modify the branching process:
  accepting $:=$ accepting $\wedge$ 1-safe
- compute the 1-safe states in that process

# The Qualitative Problem

Generalise the notion of "decent":

a tree is $k$-decent $\overset{\text{def}}{\iff}$
each branch has $\geq k$ accepting nodes (besides the root)

Note: 1-decent $\equiv$ decent and $\infty$-decent $\equiv$ good

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One can compute the 1-safe states:

- remember: $X$ is 1-safe means $\Pr_X(\text{decent}) = 1$
- apply the corollary

One can compute the $k$-safe states:

- modify the branching process:
  accepting $:=$ accepting $\wedge$ $(k-1)$-safe
- compute the 1-safe states in that process

# The Qualitative Problem

1-safe $\supseteq$ 2-safe $\supseteq$ 3-safe $\supseteq \ldots$

The sequence must stabilise after $n$ iterations.

Hence:
state $X$ is $n$-safe $\iff$ $X$ is $\infty$-safe $\iff$ $\Pr_X(\text{good}) = 1$

**Theorem (Qualitative Problem)**

*One can decide in polynomial time whether* $\Pr_X(\mathit{good}) = 1$.

# The Quantitative Problem

**Theorem (Quantitative Problem)**

*One can "efficiently approximate"* $\Pr_X(\textit{good})$.

Proof sketch. The following algorithm works:

1. compute the $\infty$-safe states in polynomial time (that is the qualitative problem).
2. modify the branching process: accepting $:=$ $\infty$-safe
3. approximate $\Pr_X(\text{decent})$ for the resulting process

Correctness is not obvious, but not hard. □

# Properties

deterministic parity (i.e., any $\omega$-regular) property along all branches

↑ generalisation: not easy

in this talk: deterministic Büchi property along all branches

# Properties

in the paper: deterministic parity tree property

↑ generalisation: easy

deterministic parity (i.e., any $\omega$-regular) property along all branches

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in this talk: deterministic Büchi property along all branches

# Logic Specifications

For Markov Chains: PCTL formulae

e.g. $[\varphi \mathsf{U} \psi]_{\geq 0.9} \;\equiv\; \Pr(\text{run satisfies } \varphi \mathsf{U} \psi) \geq 0.9$

For Branching Processes: PTTL formulae

e.g. $[\varphi \mathsf{EU} \psi]_{\geq 0.9} \;\equiv\; \Pr(\text{tree has branch satisfying } \varphi \mathsf{U} \psi) \geq 0.9$

e.g. $[\varphi \mathsf{AU} \psi]_{\geq 0.9} \;\equiv\; \Pr(\text{all branches satisfy } \varphi \mathsf{U} \psi) \geq 0.9$

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**Theorem**

*Model checking branching processes*

- *against PTTL is in PSPACE.*
- *against the qualitative fragment of PTTL is in P.*

# Properties

in the paper: deterministic parity tree property

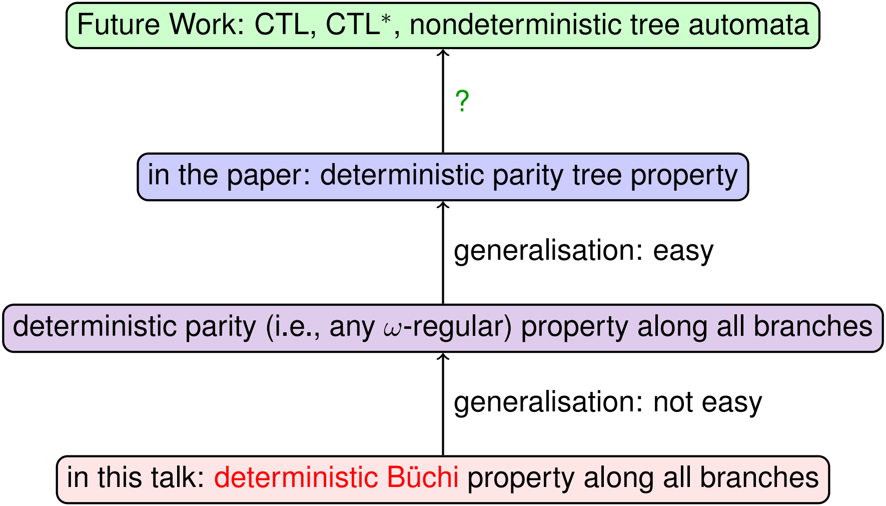
↑ generalisation: easy

deterministic parity (i.e., any $\omega$-regular) property along all branches

↑ generalisation: not easy

in this talk: deterministic Büchi property along all branches

# Properties


Future Work: CTL, CTL*, nondeterministic tree automata
?
in the paper: deterministic parity tree property
generalisation: easy
deterministic parity (i.e., any $\omega$-regular) property along all branches
generalisation: not easy
in this talk: deterministic Büchi property along all branches