On Probabilistic Parallel Programs with Process Creation and Synchronisation

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TACAS, Saarbrücken
30 March 2011

Split-Join Systems: A Model for Programs with Process Spawning

The rules of a SJS:

$$\text{split: } X \hookrightarrow \langle XX \rangle$$

$$\text{intern: } X \hookrightarrow q$$

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An example run:

$\boxed{X}$

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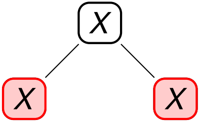

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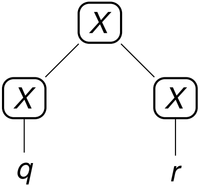

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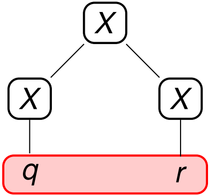

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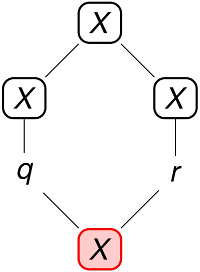

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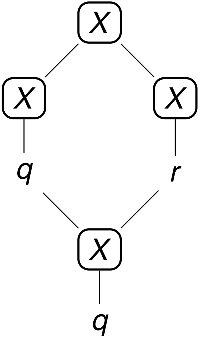

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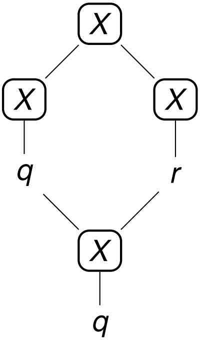

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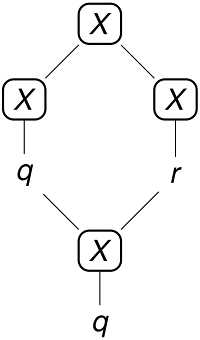

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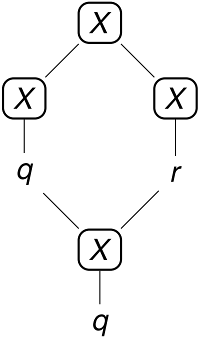

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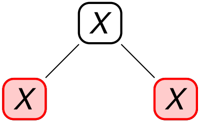

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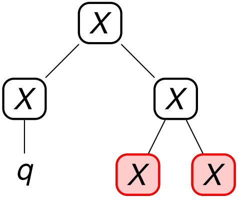

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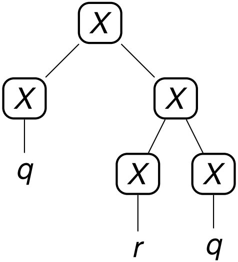

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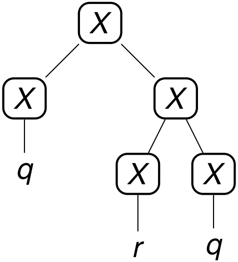

Time $T = 3$, Work $W = 5$, Space $S = 3$

Why the Sibling Constraint?

The sibling constraint prevents arbitrary synchronisation.

- Arbitrary Synchronisation would lead to Petri-Nets.
  $\longrightarrow$ more difficult to analyse
- Sibling Synchronisation is enough for modeling purposes.

Relationship to pPDSs

pSJSs subsume (probabilistic) pushdown systems (pPDSs).

Pushdown rules:

$$\begin{aligned} \text{push: } qX &\hookrightarrow rYZ \\ \text{intern: } rY &\hookrightarrow sW \\ \text{pop: } sW &\hookrightarrow t \\ \text{pop: } tZ &\hookrightarrow u \end{aligned}$$

Run $qX \Rightarrow rYZ \Rightarrow sWZ \Rightarrow tZ \Rightarrow u$

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View this run as:

$\boxed{qX}$

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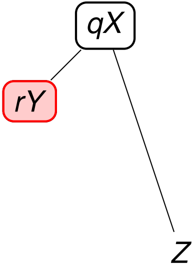

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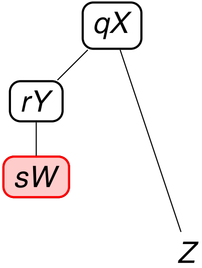

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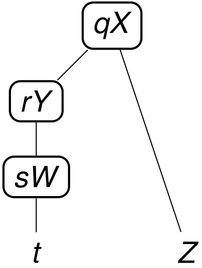

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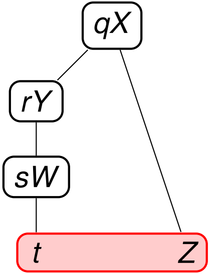

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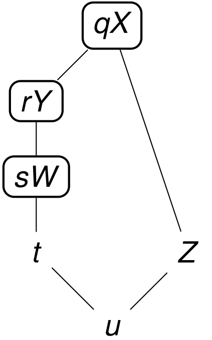

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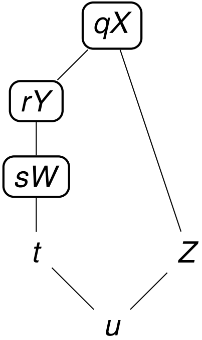

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Relationship to pPDSs

Conversely, any pSJS can be sequentialised.

The resulting pPDS is equivalent with respect to

- returned value
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Bottomline:

$\longrightarrow$ pPDSs: sequential programs
$\longrightarrow$ pSJSs: parallel programs

Relationship to Branching Processes

A branching process is a pSJS without join rules.

Example

$$X \overset{0.5}{\hookrightarrow} \langle XY \rangle \qquad\qquad Y \overset{0.3}{\hookrightarrow} X$$

$$X \overset{0.5}{\hookrightarrow} q \qquad\qquad Y \overset{0.7}{\hookrightarrow} q$$

A single synchronisation state suffices.

classical mathematical model
$\longrightarrow$ biology, physics, natural language processing, ...

pSJSs are general.

Our pSJS model generalises pPDSs and branching process.

- procedures from pPDSs
  - may be recursive
  - may return values
- parallel spawns from branching processes
- new: joins for process synchronisation and communication

Termination

split: $X \xhookrightarrow{0.5} \langle XX \rangle$

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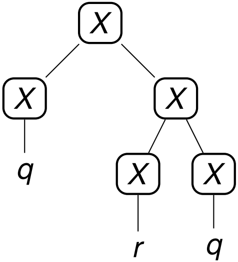
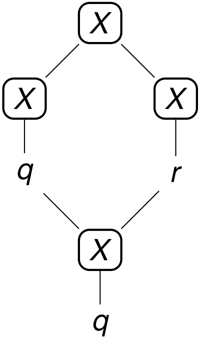
terminating runs:





One can transform the pSJS so that
all terminating runs terminate in a single state.

Termination Probability

$[X{\downarrow}q]$ = Prob. that X terminates in q

Example

$X \overset{2/3}{\hookrightarrow} \langle XX \rangle \qquad X \overset{1/3}{\hookrightarrow} q \qquad \langle qq \rangle \overset{1}{\hookrightarrow} q$

X can only terminate in q. We have $[X{\downarrow}q] = 1/2$.

Theorem

The $[X{\downarrow}q]$ are the solution of a system of polynomial equations.
Deciding whether $[X{\downarrow}q] = 0$ is in P.
Deciding whether $[X{\downarrow}q] < 1$ is PosSLP-hard even for pPDSs.

Probability of Finite Space

A pSJS may be useful even if it does not always terminate.
$\longrightarrow$ operating systems, network servers, system daemons, ...

Theorem

Let r be the probability that
a computation started in X needs only finite space.
Then r can be "efficiently expressed".
Deciding whether $r = 0$ is in P*.*
Deciding whether $r < 1$ is PosSLP*-hard even for pPDSs.*

Proof: By transforming the pSJS so that
a run terminates if and only if it needs finite space.

Applied to pPDSs, the theorem improves on [EKM05].

Work and Time

Given a pSJS and a start process X, one can compute $\mathcal{P}(W = k)$ and $\mathcal{P}(T = k)$ for $k = 1, 2, \ldots$ iteratively.

This allows to approximate $\mathbb{E}W$ and $\mathbb{E}T$.

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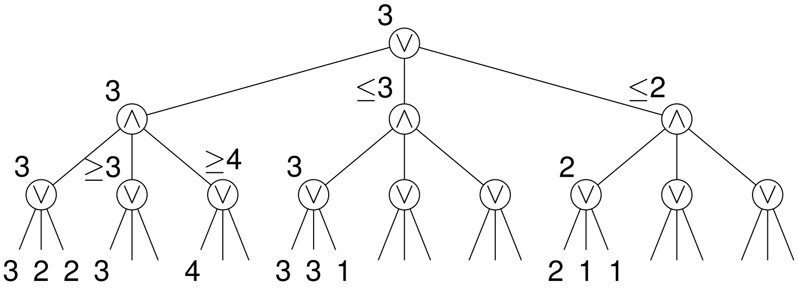
Proof sketch for the upper bound:

- transform the pSJS to a branching process
  with similar distribution of W and T (uses $[X{\downarrow}q]$)
- set up a matrix A for the branching process such that
  $A_{X,Y}$ = expected number of spawned Y-processes
  when applying an X-rule
- compute the spectral radius of A and compare with 1

Example of a Performance Analysis: Game Tree Evaluation

We wish to analyse programs that evaluate min-max trees:



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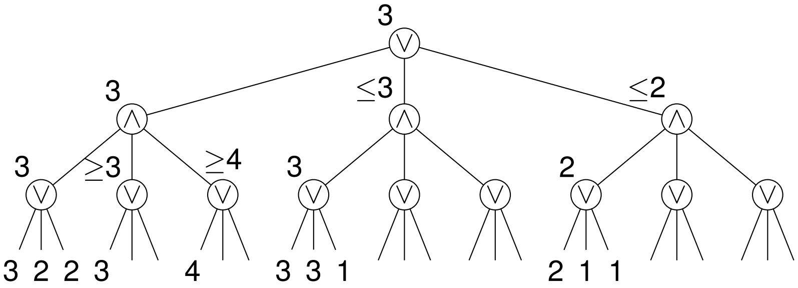

(assume 0 or 3 children per node)

A Straightforward Parallel Program

function parMax(node)
    if node.leaf() **then return** node.val()
    else parallel $\langle$ val1 := parMin(node.c1),
                                    val2 := parMin(node.c2),
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            return max$\{$val1, val2, val3$\}$

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$\longrightarrow$ parallel, but a lot of unnecessary work

(maybe deep in the tree)

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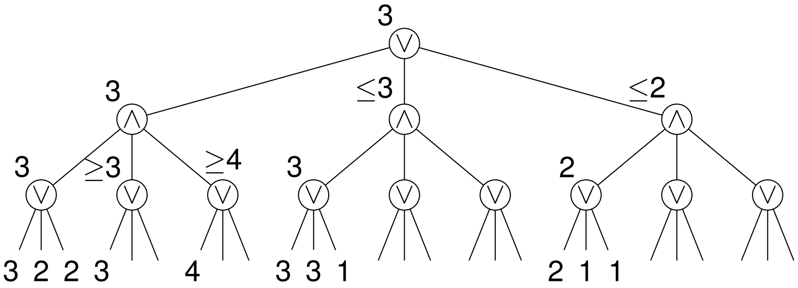

A Smart Sequential Program

```
function seqMax(node, α, β)          (initially: α = −∞, β = +∞)
  if node.leaf() then if node.val() ≤ α then return α
                      elsif node.val() ≥ β then return β
                      else return node.val()
  else val1 := seqMin(node.c1, α, β)
       if val1 = β then return β            (*cut-off after 1st child*)
       else val2 := seqMin(node.c2, val1, β)
            if val2 = β then return β       (*cut-off after 2nd child*)
            else return seqMin(node.c3, val2, β)
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$\longrightarrow$ no unnecessary work, but sequential

A Smart Parallel Program

"Young Brothers Wait":

Idea: cut-offs usually occur after the 1st child ("oldest brother")

- do the oldest brother
- only if no cut-off: do the young brothers in parallel

function YBWMax(node, α, β)
  if node.leaf() **then if** node.val() $\leq \alpha$ **then return** α
    elsif node.val() $\geq \beta$ **then return** β
    else return node.val()
  else val1 := YBWMin(node.c1, α, β)
    if val1 $= \beta$ **then return** β *(*cut-off after 1st child*)*
    else parallel $\langle$ val2 := YBWMin(node.c2, val1, β),
      val3 := YBWMin(node.c3, val1, β) $\rangle$
    return max$\{$val2, val3$\}$

Performance Analysis

These programs (par, seq, YBW) have various features:

- procedural
- recursive
- return values
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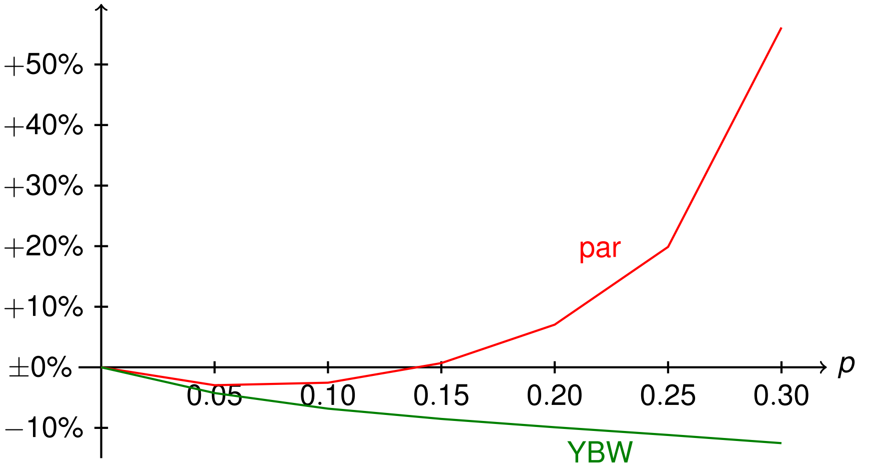
This all fits naturally in the pSJS model.

We analyse performance under prob. assumptions on the input.

Performance Analysis

Expected runtime of par and YBW compared to seq,
as a function of p (p controls the expected tree size)



Conclusions

- pSJS: new model for probabilistic parallel programs
  with process spawning and synchronisation
- Basic quantitative analysis is as expensive as for pPDSs.
- We can model, analyse and compare parallel programs
  under probabilistic assumptions.

End of Talk

Thank you!