Efficient Analysis of Probabilistic Programs with an Unbounded Counter

CAV 2011

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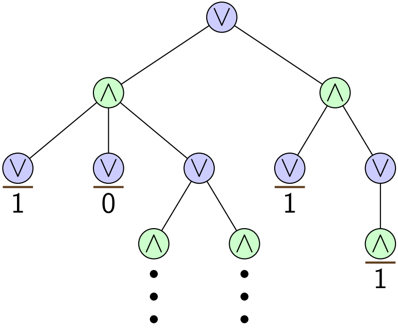
Evaluation of And-Or Trees

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procedure AND(node)
if node is a leaf
   return node.value
else
   for each successor s of node
      if OR(s) = 0 then return 0
   return 1


procedure OR(node) ...
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(evaluate only when necessary)

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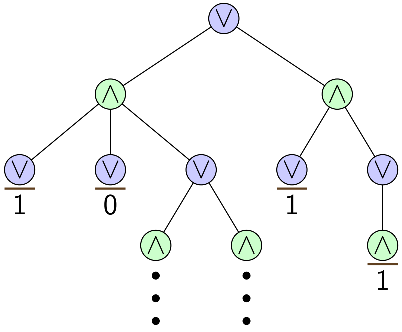


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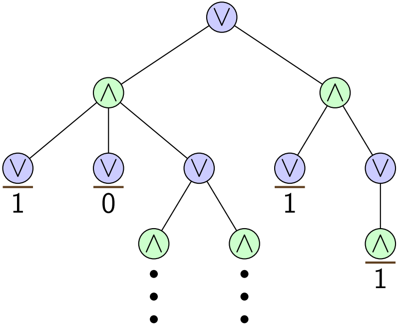


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$\Rightarrow$ probabilistic assumptions:

- AND node has 3 kids in average (geom. distribution)
- OR node has 2 kids in average
- a branch has length 4 in average
- Pr(leaf evaluates to 0) $=$ Pr(leaf evaluates to 1) $= \frac{1}{2}$

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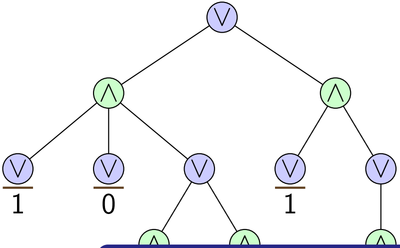


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Under these probabilistic assumptions:

Approximate efficiently the expected runtime

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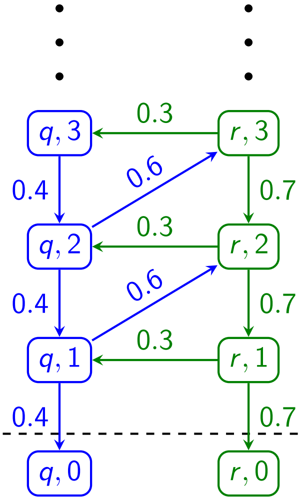
Probabilistic Counter Machines

Probabilistic Counter Machines induce infinite Markov chains:

$q \xhookrightarrow{0.6} r(+1)$ $\quad r \xhookrightarrow{0.3} q(\pm 0)$

$q \xhookrightarrow{0.4} q(-1)$ $\quad r \xhookrightarrow{0.7} r(-1)$



Modeling a Program as Prob. Counter Machine

```
procedure AND(node)
if node is a leaf
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if leaf, return 0 or 1 :

$$\mathit{and} \xhookrightarrow{\ell \cdot z} \mathit{and0}(-1)$$

$$\mathit{and} \xhookrightarrow{\ell \cdot (1-z)} \mathit{and1}(-1)$$

otherwise, call OR :

$$\mathit{and} \xhookrightarrow{1-\ell} \mathit{or}(+1)$$

if OR returns 0, return 0 immediately :

$$\mathit{or0} \xhookrightarrow{1} \mathit{and0}(-1)$$

otherwise, maybe call another OR :

$$\mathit{or1} \xhookrightarrow{x} \mathit{or}(+1)$$

$$\mathit{or1} \xhookrightarrow{1-x} \mathit{and1}(-1)$$

Applications of Probabilistic Counter Machines

PCMs model infinite-state probabilistic programs

- recursion
- unbounded data structures

PCMs = discrete-time Quasi-Birth-Death processes

- well established stochastic model
- studied since the late 60s

$\Rightarrow$ queueing theory, performance evaluation, ...

Recently: Games over (Probabilistic) Counter Machines

- energy games [Chatterjee, Doyen et al.]

$\Rightarrow$ optimizing resource consumption in portable devices

Related Model: Probabilistic Pushdown System

Probabilistic Pushdown Systems modify a stack:

$$q(X) \overset{0.3}{\hookrightarrow} r(YY)$$

$$q(X) \overset{0.5}{\hookrightarrow} r(X) \qquad q(Y) \hookrightarrow \ldots \quad r(X) \hookrightarrow \ldots \quad r(Y) \hookrightarrow \ldots$$

$$q(X) \overset{0.2}{\hookrightarrow} q(\varepsilon)$$

Prob. Pushdown Systems (equivalently, Recursive Markov Chains) are more general, but more expensive to analyze.

PCMs are Prob. Pushdown Systems with a single stack symbol.

Probabilistic Counter Machines

$q \overset{0.6}{\hookrightarrow} r(+1)$ $\quad r \overset{0.3}{\hookrightarrow} q(\pm 0)$

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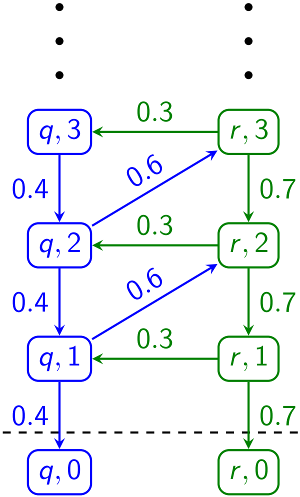

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Runtime T := number of steps from $(q, 1)$ to $(*, 0)$
We want to efficiently approximate $\mathbb{E}T$.

Trend t := "average increase of the counter per step"
Assume $t < 0$.
Intuition: The more negative the trend t, the smaller T.

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Proposition (from martingale theory: Azuma's inequality)

Let $m^{(0)}, m^{(1)}, m^{(2)}, \ldots$ be random variables with $m^{(0)} = 1$.
Let $t < 0$.
Assume $\mathbb{E}(m^{(k+1)} \mid m^{(k)}) = m^{(k)} + t$ for all k.
Then for all k: $\Pr(m^{(k)} \geq 1) \leq a^k$, where $a = e^{-t^2/2} < 1$.

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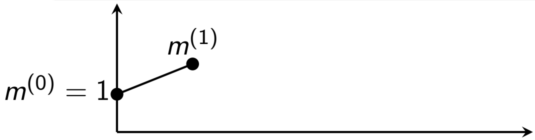

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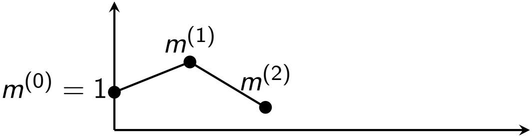

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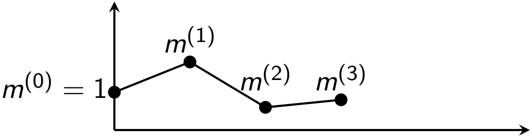

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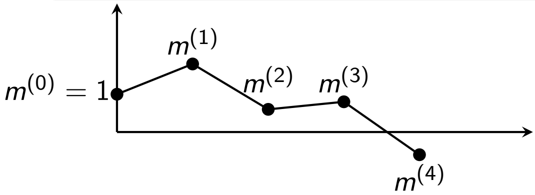

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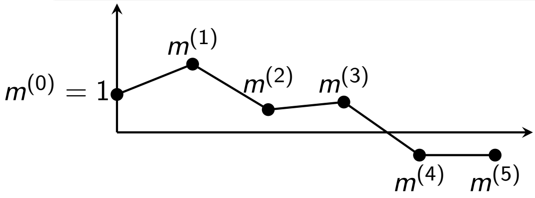

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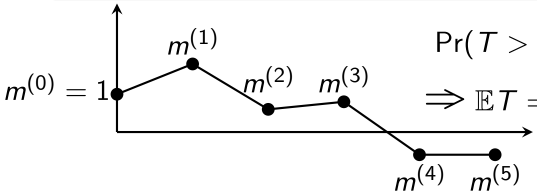

$$\Pr(T > k) \leq \Pr(m^{(k)} \geq 1) \leq a^k$$

$$\Rightarrow \mathbb{E}T = \sum_{k=0}^{\infty} \Pr(T > k) \leq \frac{1}{1-a}$$

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Let $m^{(0)}, m^{(1)}, m^{(2)}, \ldots$ be random variables with $m^{(0)} = 1$.
Let $t < 0$. **But the trend must be independent of k :-(**
Assume $\mathbb{E}(m^{(k+1)} \mid m^{(k)}) = m^{(k)} + t$ for all k.
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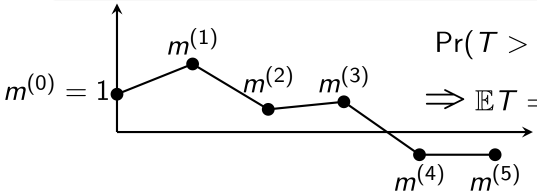

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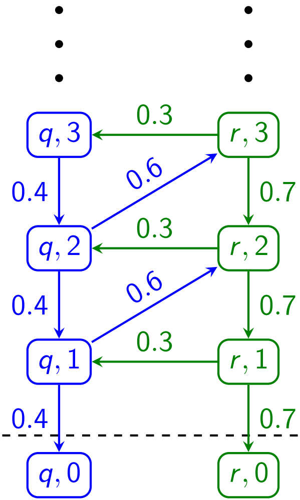

Average counter increase depends on state:

$$\begin{pmatrix} 0.4 \cdot (-1) + 0.6 \cdot (+1) \\ 0.3 \cdot 0 + 0.7 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.7 \end{pmatrix}$$

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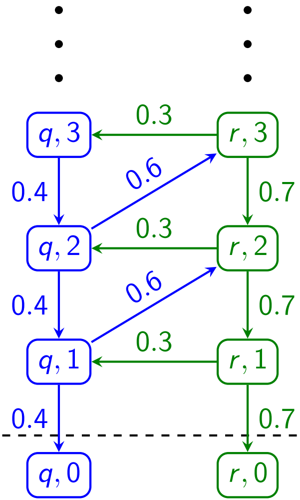

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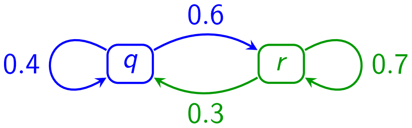

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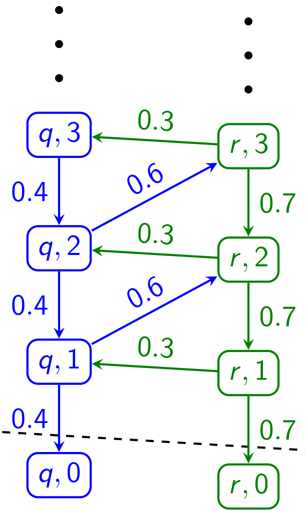

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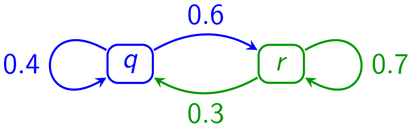

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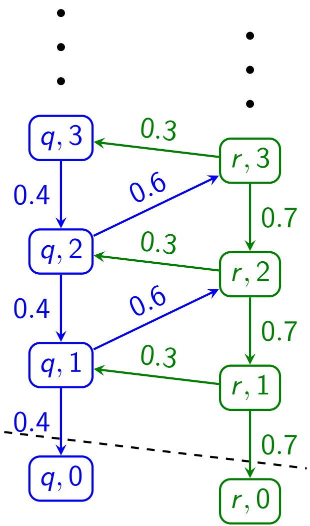

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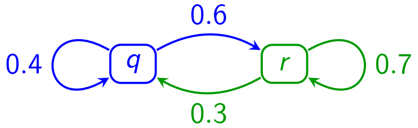

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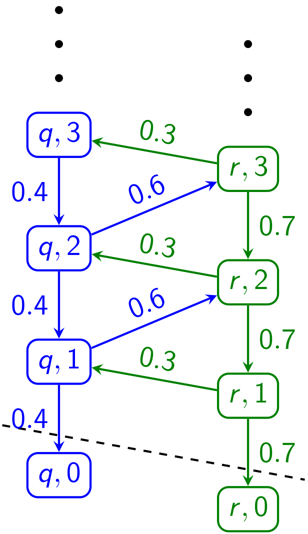

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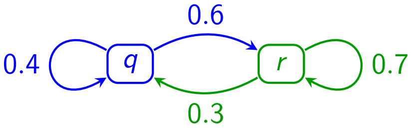

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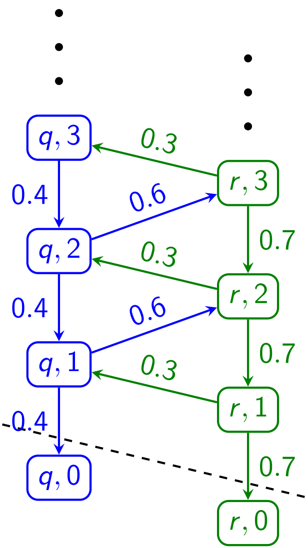

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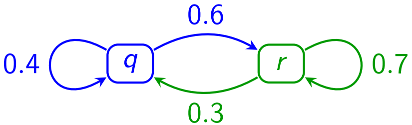

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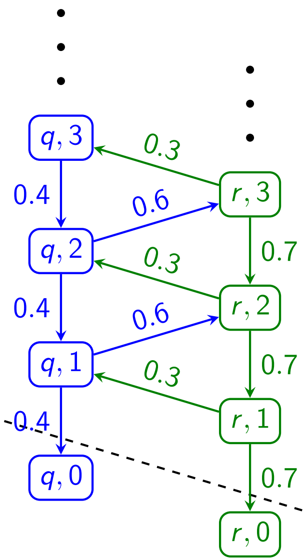

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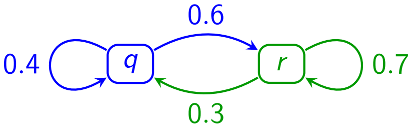

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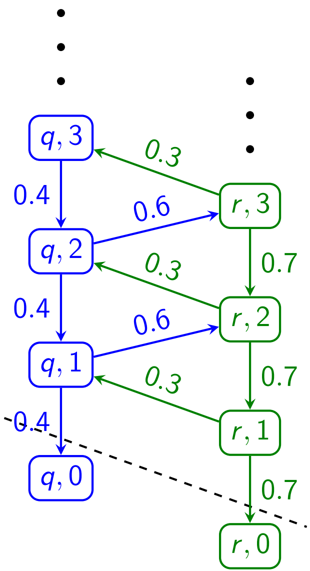

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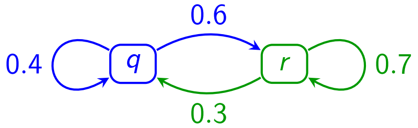

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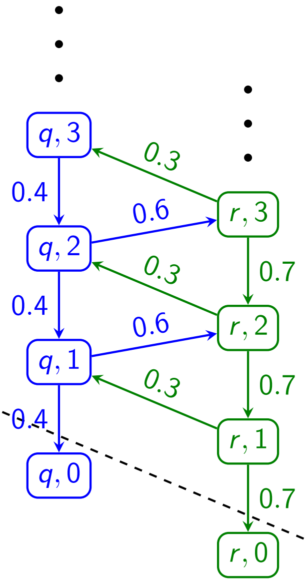

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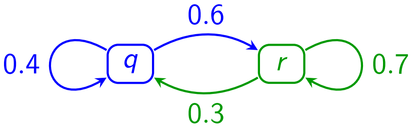

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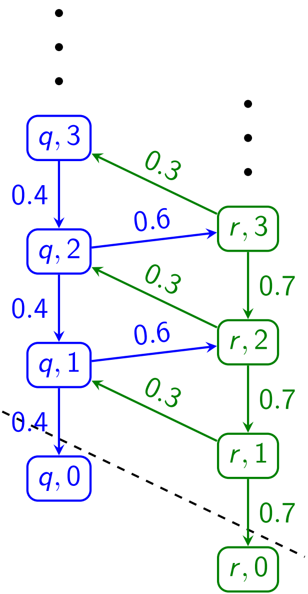

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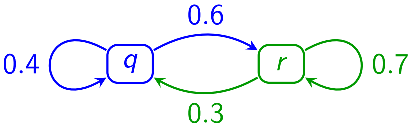

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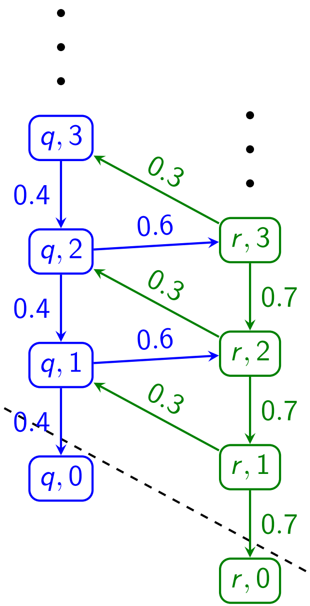

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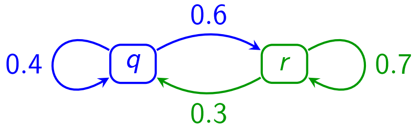
$$\begin{pmatrix} 0.4 \cdot (-1) + 0.6 \cdot (+1) \\ 0.3 \cdot 0 + 0.7 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.7 \end{pmatrix}$$

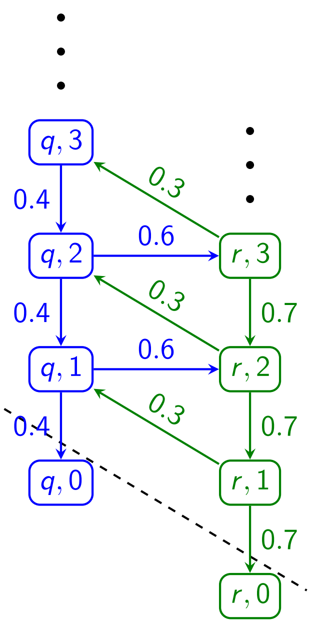
Weight this by the stationary distribution of the counterless system:



$$\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}$$

$$\text{trend } t = \left\langle \begin{pmatrix} 0.2 \\ -0.7 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \right\rangle = -0.4$$

Trend



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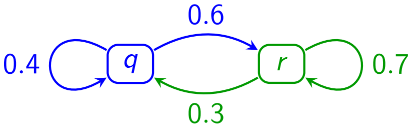

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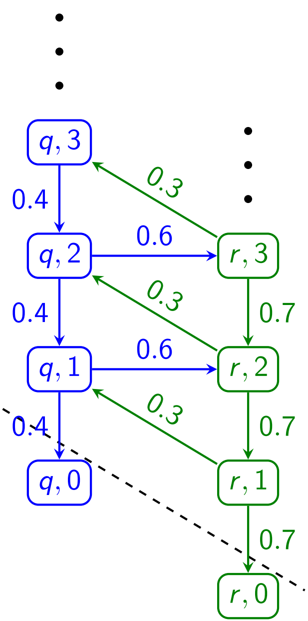

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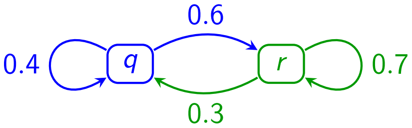

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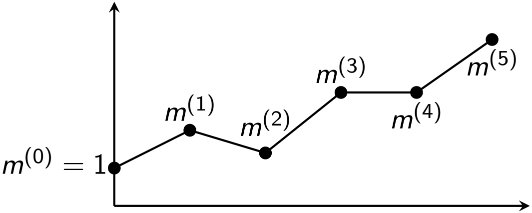
trend $t = \left\langle \begin{pmatrix} 0.2 \\ -0.7 \end{pmatrix}, \begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix} \right\rangle = -0.4$

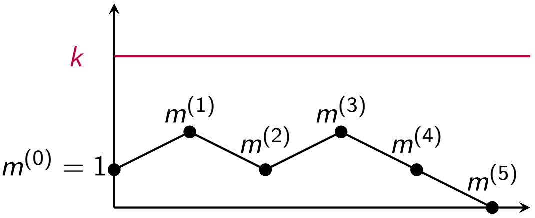
$\Longrightarrow$ expected height increase: $t = -0.4$.
independent of control state **:-)**

Positive Trend



If $t > 0$, then $\Pr(T = \infty) > 0$.
$\mathbb{E}(T \mid \text{finite})$ can be bounded as before.

Zero Trend



Proposition (from martingale theory: Optional stopping theorem)

Let $m^{(0)}, m^{(1)}, m^{(2)}, \ldots$ be random variables with $m^{(0)} = 1$.
Assume $\mathbb{E}(m^{(i+1)} \mid m^{(i)}) = m^{(i)}$ for all i.
Let $k \in \mathbb{N}$.
Let τ be the first time with $m^{(\tau)} \notin (0, k)$.
Then $\mathbb{E}m^{(\tau)} = 1$.

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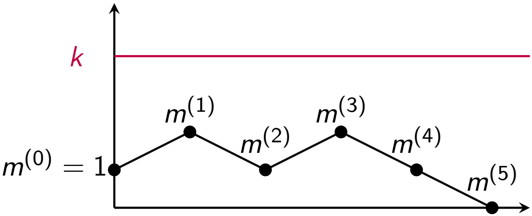

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$m^{(\tau)} = k$ $\qquad\qquad$ $m^{(\tau)} = 0$

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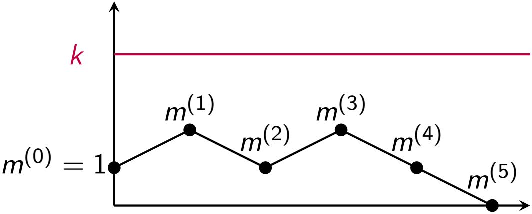

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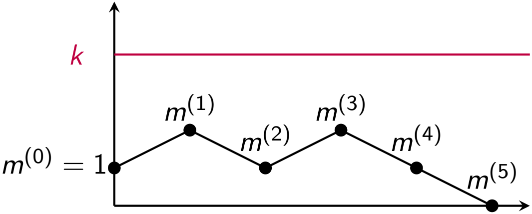

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$\implies \Pr(T \geq k) \geq \Pr(m^{(\tau)} = k) = 1/k$ and hence $\mathbb{E}\,T = \infty$

Finiteness of Expected Time

We condition on runs $q{\downarrow}r$: from $(q,1)$ reach $(r,0)$
(e.g., consider [$and{\downarrow}and0$], [$and{\downarrow}and1$])

Theorem

Either some easy case holds or one of the following:

- *If trend* $t \neq 0$, *then* $\mathbb{E}(T \mid q{\downarrow}r) \leq 85000 \cdot \dfrac{|Q|^6}{x_{\min}^{5|Q|+|Q|^3} \cdot t^4}$.
- *If trend* $t = 0$, *then* $\mathbb{E}(T \mid q{\downarrow}r)$ *is infinite.*

Corollary

Whether $\mathbb{E}(T \mid q{\downarrow}r)$ *is finite can be decided in polynomial time.*

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But we want an approximation of $\mathbb{E}(T \mid q{\downarrow}r)$.

Return Probabilities

"return probabilities" :
$[q{\downarrow}r] := \Pr\big(\text{from } (q,1) \text{ reach } (r,0)\big)$

Proposition (from [EWY'08])

- *If* $[q{\downarrow}r] > 0$, *then* $[q{\downarrow}r] \geq x_{\min}^{|Q|^3}$.
- $[q{\downarrow}r]$ *can be approximated within any error* $\varepsilon > 0$ *in time poly*$(|\mathcal{S}|, \log(1/\varepsilon))$ *in unit-cost arithmetic.*

(does not hold for pushdown systems)

Approximating Expected Runtime

Theorem

The value $\mathbb{E}(T \mid q{\downarrow}r)$ *can be approximated within any error* $\varepsilon > 0$ *in time poly*$(|\mathcal{S}|, \log(1/\varepsilon))$ *in unit-cost arithmetic.*

Use the following procedure:

- Set up an equation system $A\mathbf{x} = \mathbf{1}$. (system already known)
  Solution vector contains $\mathbb{E}(T \mid q{\downarrow}r)$ for all $q, r \in Q$.
  The matrix A contains return probabilities.
- Approximate A by approximating the return probabilities.
- Solve the approximated equation system.

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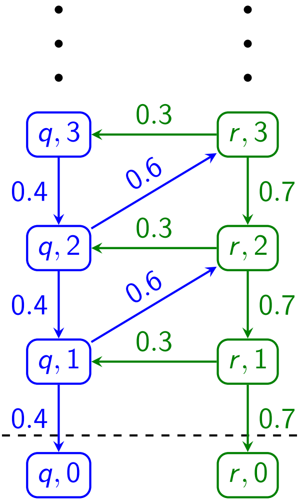
Precision of this method depends on the condition number of A.
The condition number is good enough as:

- the return probabilities cannot be too small
- the solution cannot be too large (by our bound on $\mathbb{E}T$)

Rules for Zero Counter



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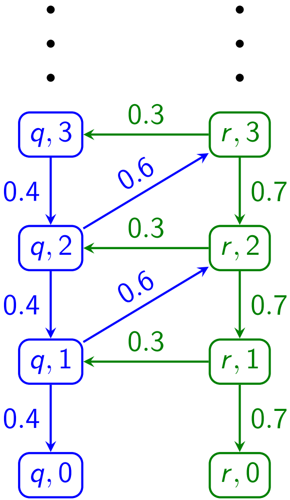

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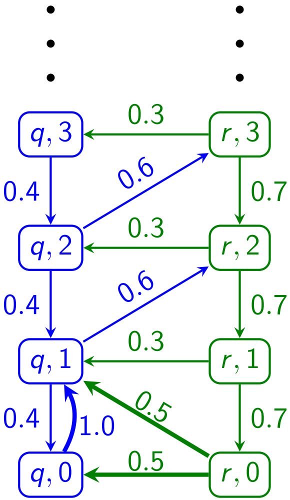

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ω-regular Specifications

Theorem

Given an ω-regular specification in terms of a Rabin automaton $\mathcal{R}$, the probability of a run satisfying the specification can be approximated within any error $\varepsilon > 0$ in time poly$(|\mathcal{S}|, |\mathcal{R}|, \log(1/\varepsilon))$ in unit-cost arithmetic.

Proof uses again "trend"-based martingale arguments.

Summary

- Probabilistic Counter Machines model infinite-state systems
  with a regular "counter-like" structure.
- Expected runtime and other quantities can be
  efficiently approximated (cf. prob. pushdown systems).
- Martingale techniques play a key role for the analysis.

Thank you!