On the Memory Consumption of Probabilistic Pushdown Automata

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Quantitative Properties of a run:

- its probability (product of the probabilities on the arrows)
- its time (number of steps till ε)
- its memory (longest configuration, “maximal stack height”)

Motivation

Probabilistic Pushdown Systems are a model for
probabilistic programs with unrestricted recursion.

Well-studied:

- model-checking for temporal logics [Etessami, Yannakakis, Esparza, Kučera, Mayr]
- long-run behaviour [Brázdil, Esparza, Kučera]
- games [Etessami, Yannakakis]

The basic quantities time and memory are random variables.

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What is the **distribution** of the memory?

Computing the Distribution of the Memory

What is the probability of reaching height ≥ 3?

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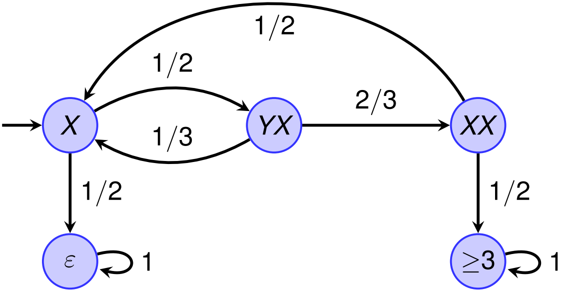
Idea: Construct the Markov chain with all stacks of height ≤ 2.



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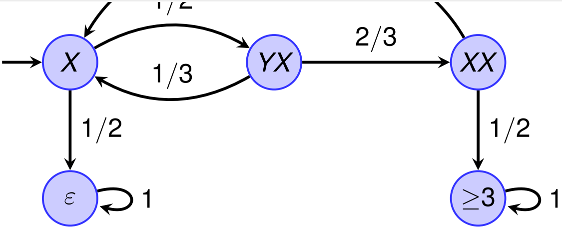

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Idea:

The Markov chain has a regular structure.
Exploit that.

Linear Equation Systems for the Distribution

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Let $\boldsymbol{p}[k]_X :=$ probability of reaching height $\geq k$ if starting with X.
Compute $\boldsymbol{p}[k]_X$ by solving linear equations. For instance:

$$\boldsymbol{p}[10]_X = 1/2 \cdot (\boldsymbol{p}[9]_Y + \qquad \boldsymbol{p}[10]_X) + 1/2 \cdot 0$$

$$\boldsymbol{p}[10]_Y = 2/3 \cdot \boldsymbol{p}[10]_X + 1/3 \cdot 0$$

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What about the expectation of the memory consumption?

In the following we assume finite expectation of the memory (which is the most important case).

Putting the Equations in Matrix Form

After some (non-trivial but non-interesting) normalizations:

There is a (nonnegative) matrix $A(\boldsymbol{x}) \in \mathbb{R}^{\Gamma \times \Gamma}$
that depends on a (nonnegative) vector $\boldsymbol{x} \in \mathbb{R}^{\Gamma}$ such that:

- $A(\boldsymbol{x})$ increases monotonically with $\boldsymbol{x}$,
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The sequence $\boldsymbol{t}[1] \leq \boldsymbol{t}[2] \leq \ldots$ converges to a limit $\boldsymbol{t}$,
where $\boldsymbol{t}$ is the vector of termination probabilities.

By monotonicity and continuity, the sequence
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It follows:

$$\boldsymbol{p}[k+1] = A(\boldsymbol{t}[k]) \cdot \boldsymbol{p}[k] \leq A \cdot \boldsymbol{p}[k] \leq A^k \cdot \boldsymbol{p}[1] = A^k \cdot \mathbf{1}$$

Approximating the Distribution

On the previous slide: $\boldsymbol{p}[k] \leq A^{k-1} \cdot \mathbf{1}$,
so $A^{k-1} \cdot \mathbf{1}$ is an upper bound on $\boldsymbol{p}[k]$.

Advantage: compute A^{k-1} by repeated squaring: $A, A^2, A^4, \ldots$
$\Longrightarrow$ only $\mathcal{O}(\log k \cdot |\Gamma|^3)$ operations
$\Longrightarrow$ safe overapproximation for large k

How tight is the overapproximation?

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There is a real number d with $0 < d \leq 1$ such that for all k:

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Theorem

We have $\rho < 1$. The proposition above implies:

$$\boldsymbol{p}[k] \in \Theta(\rho^k)$$

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EM can be (under-)approximated by $UM[\ell] := \sum_{k=1}^{\ell} \boldsymbol{p}[k]_X$.

Theorem

The sequence $(UM[\ell])_\ell$ *converges to EM.*
More precisely, one can compute $a > 0$ *and* $0 < b < 1$ *with*

$$EM - UM[\ell] \leq a \cdot b^{\ell} \ ,$$

so the error decays exponentially.

Finiteness of the Expectation

Most presented results hold if the expectation is finite.
How to decide whether the expectation is finite?

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Theorem

Whether EM is finite can be decided

- *in polynomial time for pushdown systems without states*
- *in polynomial space for general pushdown systems.*

The problem is PosSLP-hard (therefore unlikely in P)
for general pushdown systems.

Conclusions

- Probabilistic pushdown systems model
  recursive probabilistic programs.
- We studied one basic random variable, the memory.
- Its distribution can be computed
  - naively (using an exponential-sized Markov chain),
  - by solving linear equation systems,
  - by efficiently computing overapproximations of the memory.
- The expectation can be approximated,
  and the error decays exponentially.
- Whether the expectation is finite can be decided in
  polynomial time for pushdown systems without states.
- Open question: How to decide whether
  the expectation exceeds a given bound?

End of Talk

Thank you!