Verification of Infinite-state Systems

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Software model checking

Challenge: develop model-checking techniques for ‘higher-level’ software.

Three main research questions:

- Integration of the techniques in the system development process.
- Checking Lucent’s PathStar access server.
- Checking Windows XP drivers.
- Automatic extraction of formal models from code.
- Work of the abstract interpretation and static analysis community.
- From Java code to model-checkable models through abstraction/static analysis.
- Exploration of infinite-state spaces.
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Exploration of infinite-state spaces.
Integration in the system development process

PathStar
Checking a telephone switch.
- One system
- Verification interleaved with design (300 versions)
- Highly concurrent code
- Complex specification (80/200 properties)

Slam
Checking Windows XP drivers.
- Many systems
- Post-mortem verification
- Sequential code
- Simple specification (i.e., correct locking/unlocking)
Sources of infinity in software systems

Data manipulation: integers, lists, trees, more general pointer structures, . . .

Control structures: procedures, process creation, . . .

Asynchronous communication: unbounded FIFO queues.

Parameters: number of processes, duration of delays . . .

Real-time: discrete or dense domains.
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Model data abstractions of the program by means of extended automata or equivalent models.
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Using the automata theoretic-approach to model checking, reduce the verification problem to reachability or repeated reachability problems. (See Moshe Vardi’s course.)
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Develop algorithms or semi-algorithms for these problems using symbolic search, accelerations, and learning. (See this course.)

Reintroduce the abstracted data incrementally by means of predicate abstraction and counterexample-guided abstraction refinement. (See Orna Grumberg’s course.)
Extended automata: Syntax

Extended automaton = automaton whose transitions are guarded by and operate on data structures.

An extended automaton is a tuple \( E = (X, Q, T, G, A) \) where

- \( X = \{ x_1, \ldots, x_n \} \) is a finite set of variables over sets \( V_1, \ldots, V_n \) of values,
- \( Q \) is a finite set of control states,
- \( T \subseteq Q \times Q \) is a set of transitions or rules,
- \( G \) associates to each transition a guard (a predicate over \( X \), the condition under which the transition can be taken),
- \( A \) associates to each transition an action (a possibly nondeterministic assignment to \( X \))

Notation for transitions: \( q \xrightarrow{g,a} q' \), where \( g \) guard and \( a \) action.

Remark: variables over finite sets of values can be encoded into the states.
A configuration is a tuple $\langle q, v_1, \ldots, v_n \rangle$, where

- $q$ is a state, and
- $v_1, \ldots, v_n$ is a valuation of $x_1, \ldots, x_n$ (i.e., $v_i \in V_i$ for every $1 \leq i \leq n$).

The transition system $\mathcal{T}_E$ of an extended automaton $E$ has:

- the set of all configurations as nodes, and
- an edge $\langle q, v_1, \ldots, v_n \rangle \xrightarrow{a} \langle q', v'_1, \ldots, v'_n \rangle$ iff $E$ has a transition $q \xrightarrow{a} q'$ such that
  - $v_1, \ldots, v_n$ satisfies the guard $g$, and
  - $v'_1, \ldots, v'_n$ is one of the possible results of applying $a$ to $v_1, \ldots, v_n$. 
Some classes of extended automata

<table>
<thead>
<tr>
<th>Automata</th>
<th>Variables</th>
<th>Transition</th>
</tr>
</thead>
<tbody>
<tr>
<td>Timed automata</td>
<td>clocks (reals)</td>
<td>( q \xrightarrow{c_1 &gt; 2} c_2 := 0 \rightarrow q' )</td>
</tr>
<tr>
<td>Pushdown automata</td>
<td>stack</td>
<td>( q \xrightarrow{\text{top} = a} a/\text{ba} \rightarrow q' )</td>
</tr>
<tr>
<td>(Ext. of) Petri nets</td>
<td>counters (integers)</td>
<td>( q \xrightarrow{x_1 = 0} x_2 := x_2 + x_3 \rightarrow q' )</td>
</tr>
<tr>
<td>FIFO automata</td>
<td>queues</td>
<td>( q \xrightarrow{l_1 \neq \epsilon} l_2 ? a \rightarrow q' )</td>
</tr>
</tbody>
</table>
A network of extended automata (or just a network) is a tuple $\langle E_1, \ldots, E_m \rangle$ of extended automata over the same set of variables $X$.

The asynchronous product of a network $\langle E_1, \ldots, E_m \rangle$ is the extended automaton having

- the set $Q = Q_1 \times \ldots \times Q_m$ as states, where $Q_1, \ldots, Q_m$ are the sets of states of $E_1, \ldots, E_m$, and
- for every $i \in \{1, \ldots, m\}$, every state $\langle q_1, q_2, \ldots, q_m \rangle \in Q$ and every transition $q_i \xrightarrow{a} q_i'$ of $E_i$, a transition $\langle q_1, \ldots, q_{i-1}, q_i, q_{i+1}, \ldots, q_m \rangle \xrightarrow{a} \langle q_1, \ldots, q_{i-1}, q_i', q_{i+1}, \ldots, q_m \rangle$
The reachability problem

Let $c, c'$ be two configurations of an extended automaton $E$. We say that $c'$ is reachable from $c$ if there is a path in $\mathcal{T}_E$ leading from $c$ to $c'$.

We consider the following problem:

- **Given**: An extended automaton $E$, a set $I$ of initial configurations, a set $D$ of dangerous configurations.
- **Decide**: Is some dangerous configuration reachable from some initial configuration?

The sets $I$ and $D$ may be infinite.
Symbolic search

A general framework for the reachability problem

Let $post(C)$ denote the immediate successors of a (possibly infinite!) set $C$ of configurations

Forward symbolic search

Initialize $C := I$

Iterate $C := C \cup post(C)$ until

$C \cap D \neq \emptyset$; return “reachable”, or

a fixpoint is reached; return “non-reachable”

Backward search: exchange $I$ and $D$, replace $post$ by $pre$.

Question: when is symbolic search effective?
(Forward) Symbolic search effective if . . .

1. each $C \in C$ has a symbolic finite representation,

2. $I \in C$,

3. if $C \in C$, then $C \cup \text{post}(C) \in C$ (and effectively computable),

4. emptiness of $C \cap D$ is decidable,

5. $C_1 = C_2$ is decidable (to check if fixpoint has been reached), and

6. any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ reaches a fixpoint after finitely many steps.
Remarks

Similar conditions for backward search.

The shape of $I$ is determined by the model.

The shape of $D$ is determined by the specification.

This asymmetry can make one of the two searches far more useful than the other.
Program for the rest of the course

We consider four classes of systems, and use them to illustrate four different techniques to obtain an effective symbolic search.

- **Timed automata**: Finite partitions.
- **Broadcast protocols**: Well quasi-orders.
- **Pushdown automata**: Accelerations.
- **(Lossy) channel systems**: Learning.
Timed automata
Timed automata

Automata extended with clocks (non-negative real variables).

Time-elapse transitions: self-loops, no guard, the action adds an arbitrary positive real to all clocks (same for all).

Location-switch transitions: guarded by boolean combination of comparisons with integer bounds, the action resets a subset of clocks.
Timed automata

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Case study: Fischer’s mutex protocol

A simplified version (so that the analysis can be visualized in one slide).

```plaintext
var v:\{1,2\} init 1;

\text{delay} < 1;
\text{delay} > 1;
\text{if } v = 1 \text{ then goto cs1}
```

```plaintext
\text{delay} < 1;
\text{delay} > 1;
\text{if } v = 2 \text{ then goto cs2}
```
\textbf{Model}

\texttt{var v : \{1, 2\} init 1}
\texttt{var c_1, c_2 : clock init 0}

\begin{tikzpicture}[node distance = 2cm, auto]
\node (A1) [circle, draw] {$A_1$};
\node (B1) [circle, draw, right of=A1] {$B_1$};
\node (CS1) [circle, draw, right of=B1] {$CS_1$};
\node (A2) [circle, draw, below of=A1] {$A_2$};
\node (B2) [circle, draw, below of=B1] {$B_2$};
\node (CS2) [circle, draw, below of=CS1] {$CS_2$};

\path[->]
(A1) edge node {$c_1 < 1$
\texttt{v := 1, c_1 := 0}} (B1)
(B1) edge node {$c_1 > 1 \land v = 1$} (CS1)
(A2) edge node {$c_2 < 1$
\texttt{v := 2, c_2 := 0}} (B2)
(B2) edge node {$c_2 > 1 \land v = 2$} (CS2);
\end{tikzpicture}

Network of 2 timed automata.

Equivalent to one single automaton with 9 states.
Symbolic search for timed automata

The set $I$ of initial configurations is usually of the form

$$\{\langle q, 0, \ldots, 0 \rangle \mid q \in Q_I\}$$

The set $D$ of dangerous final configurations is usually of the form

$$\{\langle q, t_1, \ldots, t_n \rangle \mid q \in Q_D \text{ and } t_1, \ldots, t_n \geq 0\}$$

Question: Is reachability decidable for $I$ and $D$ of this form?
Regions

Consider a timed automaton with clocks $x_1, \ldots, x_n$.

Let $\max$ be the maximal constant appearing in the syntactic description of the automaton.
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Let $\text{max}$ be the maximal constant appearing in the syntactic description of the automaton.

Let $\Gamma$ be the set of all constraints of the form

$$x_i \leq k \quad \text{or} \quad x_i \geq k \quad \text{or} \quad x_i - x_j \leq k$$

where $k \in \{0, 1, \ldots, \text{max}\}$. 

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Two configurations $\langle q, t \rangle$ and $\langle r, u \rangle$ are equivalent, denoted by $\langle q, t \rangle \sim \langle r, u \rangle$, if

- $q = r$, and
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An equivalence class of configurations is called a region.
Characterizing regions

Given a real number $z$, let $\lfloor z \rfloor$ denote its integer and $z$ its fractional part.

$\langle q, t \rangle \sim \langle r, u \rangle$ holds iff $q = r$ and for every $i, j \in \{0, 1, \ldots, max\}$:

(a) $\lfloor t_i \rfloor = \lfloor u_i \rfloor$ or $t_i > \text{max}$ and $u_i > \text{max}$,

(because $k - 1 \leq t_i \leq k$ iff $k - 1 \leq u_i \leq k$ for all $k \in \{1, \ldots, \text{max}\}$)

(b) if $t_i, u_i \leq \text{max}$, then $t_i = 0$ iff $u_i = 0$,

(because $k \leq t_i \leq k$ iff $k \leq u_i \leq k$ for all $k \in \{0, \ldots, \text{max}\}$)

(c) if $t_i, u_i, t_j, u_j \leq \text{max}$, then $t_i < t_j$ iff $u_i < u_j$.

(because of (a), (b), and $t_i - t_j \leq 0$ iff $u_i - u_j \leq 0$)

Example: $\langle q \; 3.2 \; 4.7 \; 3.5 \rangle \sim \langle q \; 3.7 \; 4.9 \; 3.8 \rangle$

$\langle q \; 3.2 \; 4.7 \; 3.5 \rangle \not\sim \langle q \; 3.2 \; 4.7 \; 3.9 \rangle$
Two observations

The number of regions is bounded by $(2^{\text{max}} + 2)^n \cdot n! \cdot 2^n$ (exercise).

- Exponential in both the number of clocks $n$ and in the maximal constant $\text{max}$ when written in binary.
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Two equivalent configurations enable exactly the same transitions.

- Because they satisfy exactly the same guards.
Effectiveness of forward and backward search

We choose $C$ as the powerset of the set of regions.

Theorem [Alur, Dill, TCS 1994]:
Both forward and backward search satisfy conditions (1) - (6).

Proof for forward search in the next slides, for backward search analogous.
Proof

1. A region can be finitely represented by the set of constraints it satisfies (by definition).
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$(0, \ldots, 0)$ is the only time-vector satisfying $x_i \leq 0$ for $i \in \{1, \ldots, n\}$, and so $\{\langle q, 0, \ldots, 0 \rangle \}$ is a region for each state $q$. 

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3. If $C$ is the union of a set of regions, then so is $C \cup \text{post}(C)$.

It suffices to prove that if $C$ is a region then $\text{post}(C)$ is a union of regions.

Take $\langle r, u \rangle \in \text{post}(C)$ and $\langle r, u' \rangle \sim \langle r, u \rangle$. We show $\langle r, u' \rangle \in \text{post}(C)$.

Since $\langle r, u \rangle \in \text{post}(C)$, there is $\langle q, t \rangle \in C$ such that $\langle q, t \rangle \rightarrow \langle r, u \rangle$.

We consider the cases of time-elapse and location-switch transitions separately.
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Time-elapse transitions ("proof by example"): 

\[ \langle q, t_1, t_2, t_3 \rangle \xrightarrow{|\tau| + \tau} \langle r, u_1, u_2, u_3 \rangle \]

\[ \sim \]

\[ \langle r, u'_1, u'_2, u'_3 \rangle \]
Time-elapse transitions ("proof by example"):

\[
0 < t_1 < t_2 < t_3 < 1
\]

\[
\langle q \ t_1 \ t_2 \ t_3 \rangle \xrightarrow{\lfloor \tau \rfloor + \tau} \langle r \ u_1 \ u_2 \ u_3 \rangle
\]

\[
\sim
\]

\[
\langle r \ u'_1 \ u'_2 \ u'_3 \rangle
\]

\[
0 = u_2 < u_3 < u_1 < 1
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Time-elapse transitions ("proof by example"): 

\[ \begin{align*} 
0 < t_1 < t_2 < t_3 < 1 & \quad \text{and} \quad 0 = u_2 < u_3 < u_1 < 1 \\
\langle q, t_1, t_2, t_3 \rangle & \xrightarrow{[\tau] + \tau} \langle r, u_1, u_2, u_3 \rangle \\
\langle q, 3.1, 1.5, 2.7 \rangle & \xrightarrow{1 + 0.5} \langle r, 4.7, 3.0, 4.3 \rangle \\
\sim & \\
\langle r, u'_1, u'_2, u'_3 \rangle 
\end{align*} \]
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\[ \langle q \ 3.1 \ 1.5 \ 2.7 \rangle \xrightarrow{1 + 0.5} \langle r \ 4.7 \ 3.0 \ 4.3 \rangle \]
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Time-elapse transitions ("proof by example"):

\[ 0 < t_1 < t_2 < t_3 < 1 \]
\[ \langle q \ t_1 \ t_2 \ t_3 \rangle \xrightarrow{\tau} \langle r \ u_1 \ u_2 \ u_3 \rangle \]
\[ \langle q \ 3.1 \ 1.5 \ 2.7 \rangle \xrightarrow{1 + 0.5} \langle r \ 4.7 \ 3.0 \ 4.3 \rangle \]
\[ \sim \]
\[ \langle r \ 4.8 \ 3.0 \ 4.7 \rangle \]
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\]
\[
\sim
\]
\[
\langle q \ t'_1 \ t'_2 \ t'_3 \rangle \xrightarrow{\lfloor \tau \rfloor + \delta} \langle r \ u'_1 \ u'_2 \ u'_3 \rangle
\]
\[
0 < t'_1 < t'_2 < t'_3 < 1
\]
\[
u'_3 < \delta < u'_1
\]
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\begin{align*}
0 < t_1 < t_2 < t_3 < 1 & \quad \quad 0 = u_2 < u_3 < u_1 < 1 \\
\langle q, t_1, t_2, t_3 \rangle & \quad \quad \langle r, u_1, u_2, u_3 \rangle \\
\langle q, 3.1, 1.5, 2.7 \rangle & \quad \quad \langle r, 4.7, 3.0, 4.3 \rangle \\
\sim & \quad \sim \\
\langle q, 3.05, 1.25, 2.95 \rangle & \quad \quad \langle r, 4.8, 3.0, 4.7 \rangle \\
\langle q, t'_1, t'_2, t'_3 \rangle & \quad \quad \langle r, u'_1, u'_2, u'_3 \rangle \\
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\langle q & \quad 3.1 \quad 1.5 \quad 2.7 \rangle & \quad \rightarrow & \quad \langle r & \quad 4.7 \quad 3.0 \quad 4.3 \rangle \\
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Location-switch transitions (“proof by example”):

\[ \langle r \ t_1 \ t_2 \ t_3 \rangle \xrightarrow{x_2 := 0} \langle r \ t_1 \ 0 \ t_3 \rangle \]

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0 < t_1 < t_2 < t_3 < 1 \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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\quad \quad \quad \quad \quad \quad }
Location-switch transitions ("proof by example"): 

\[
0 < t_1 < t_2 < t_3 < 1
\]

\[
\langle r \ t_1 \ t_2 \ t_3 \rangle \xrightarrow{x_2 := 0} \langle r \ 3.1 \ 1.5 \ 2.7 \rangle
\]

\[
0 = u_2 < u_1 < u_3 < 1
\]
Location-switch transitions ("proof by example"):

\[
\begin{align*}
0 < t_1 < t_2 < t_3 < 1 & \quad 0 = u_2 < u_1 < u_3 < 1 \\
\langle r \ t_1 \ t_2 \ t_3 \rangle & \xrightarrow{x_2 := 0} \langle r \ 3.1 \ 1.5 \ 2.7 \rangle \\
\langle r \ 3.1 \ 1.5 \ 2.7 \rangle & \xrightarrow{x_2 := 0} \langle r \ 3.1 \ 0 \ 2.7 \rangle \\
\sim & \\
\langle r \ 3.3 \ 0 \ 2.4 \rangle & \\
\langle r \ u'_1 \ 0 \ u'_3 \rangle & \quad 0 = u'_2 < u'_1 < u'_3 < 1
\end{align*}
\]
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0 < t_1 < t_2 < t_3 < 1
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\[
\sim
\]
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\langle r & \ t'_1 \ t'_2 \ t'_3 \ \rangle \xrightarrow{x_2 := 0} \langle r & \ 3.3 \ 0 \ 2.4 \ \rangle \\
0 < t'_1 = u'_1 < t'_2 < t'_3 = u'_3 < 1
\end{align*}
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0 = u'_2 < u'_1 < u'_3 < 1
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\[0 < t_1 < t_2 < t_3 < 1\] 
\[\langle r \ t_1 \ t_2 \ t_3 \rangle \quad \xrightarrow{x_2 := 0} \quad \langle r \ t_1' \ 0 \ t_3 \rangle\] 
\[\langle r \ 3.1 \ 1.5 \ 2.7 \rangle \quad \xrightarrow{x_2 := 0} \quad \langle r \ 3.1 \ 0 \ 2.7 \rangle\] 
\[\sim\] 
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\[
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0 < t'_1 = u'_1 < t'_2 < t'_3 = u'_3 < 1 \\
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Just check if $C$ contains some configuration with some state of $Q_D$ as first element.
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A region is represented by the constraints it satisfies.
Two regions are equal iff their representations are equal.
Two sets of regions are equal iff they contain the same regions.
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6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.
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6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.

Follows from the fact that the set of regions is finite.
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   Just check if $C$ contains some configuration with some state of $Q_D$ as first element.

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   Two sets of regions are equal iff they contain the same regions.

6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.

   Follows from the fact that the set of regions is finite.
(One half of) The region graph of Fischer’s protocol

A₁, A₂, v = 1
c₁ = c₂ = 0

A₁, A₂, v = 1
0 < c₁ = c₂ < 1

A₁, A₂, v = 1
c₁ = c₂ = 1

A₁, A₂, v = 1
1 < c₁, c₂

A₁, B₂, v = 2
c₁ = c₂ = 0

A₁, B₂, v = 2
0 < c₁ < 1, c₂ = 0

A₁, B₂, v = 2
0 < c₂ < c₁ < 1

A₁, B₂, v = 2
0 < c₂ < c₁ = 1

A₁, B₂, v = 2
1 < c₁, c₂

A₁, B₂, v = 2
c₂ = 1 < c₁

A₁, B₂, v = 2
0 < c₂ < 1 < c₁

A₁, CS₂, v = 2
1 < c₁, c₂
The reachability problem is **PSPACE-complete**.

Reason: exponential dependence in the number of clocks or the size of max is unavoidable.

The problem remains PSPACE-hard if the constants or the number of clocks (but not both) are bounded.
Repeated reachability for timed automata

A control state is repeatedly reachable if some non-zeno infinite execution containing infinitely many location-switch transitions visits the control state infinitely often.

The repeated reachability problem can be solved easily using the region graph.
To know more


Check the publications of: Alur, Asarin, Bouyer, Courcoubetis, Dill, Henzinger, Laroussinie, Larsen, Maler, Sifakis, Wilke . . . .

UPPAAL is a popular tool for verification of timed automata, http://www.uppaal.com/
Broadcast protocols
Broadcast protocols

Introduced by Emerson and Namjoshi in LICS ’98.

All processes execute the same algorithm, i.e., all finite automata are identical.

Processes are indistinguishable (no IDs).

Communication mechanisms:

  **Rendezvous**: two processes exchange a message and move to new states.

  **Broadcasts**: a process sends a message to all others, all processes move to new states.

We introduce syntax and semantics and show translation into extended automata.
a!!  : broadcast a message along (channel) $a$

a??  : receive a broadcasted message along $a$

b!    : send a message to one process along $b$

b?    : receive a message from one process along $b$

c     : change state without communicating with anybody
The global state of a broadcast protocol is completely determined by the number of processes in each state.

**Configuration:** mapping $c : Q \rightarrow \mathbb{N}$ represented by the vector $(c(q_1), \ldots, c(q_n))$.

**Semantics for a given initial configuration:** finite transition system with configurations as nodes.
(3, 1, 2) → (4, 0, 2)  (silent move c)
(3, 1, 2) → (3, 2, 1)  (rendezvous b)
(3, 1, 2) → (2, 1, 3)  (broadcast a)
(185, 3425, 17) → (17, 1, 3609)  (broadcast a)
Parametrized configuration: partial mapping $p : Q \rightarrow \mathbb{N}$.

- Intuition: “configuration with holes”.
- Formally: set of configurations (total mappings matching $p$).

Infinite transition system of the broadcast protocol:

- Fix an initial parametrized configuration $p_0$.
- Take the union of all finite transition systems for each configuration $c \in p_0$. 
Case study: A MESI cache-coherence protocol

\[ \text{rh : read hit} \]
\[ \text{rm : read miss} \]
\[ \text{w : write hit/write miss} \]
Broadcast protocols as extended automata

We translate the MESI-protocol into an extended automaton.

We take:

- One (non-negative) integer variable per state of the protocol: $m, e, s, i$.
- One single control state $q$.
- One transition $q \xrightarrow{a} q$ for each send transition or silent move of the protocol, see next slide.

A configuration $(n_1, \ldots, n_k)$ of a broadcast protocol corresponds to the configuration $\langle q, n_1, \ldots, n_k \rangle$ of the extended automaton.
<table>
<thead>
<tr>
<th>Transition</th>
<th>Guard</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I \xrightarrow{rm!!} S$</td>
<td>$i \geq 1$</td>
<td>$m' = m \quad e' = 0 \quad s' = m + e + s + 1 \quad i' = i - 1$</td>
</tr>
<tr>
<td>$I \xrightarrow{w!!} E$</td>
<td>$i \geq 1$</td>
<td>$m' = 0 \quad e' = 1 \quad s' = 0 \quad i' = m + e + s + i - 1$</td>
</tr>
<tr>
<td>$S \xrightarrow{w!!} E$</td>
<td>$s \geq 1$</td>
<td>$m' = 0 \quad e' = 1 \quad s' = 0 \quad i' = m + e + s + i - 1$</td>
</tr>
<tr>
<td>$S \xrightarrow{rh} S$</td>
<td>$s \geq 1$</td>
<td>$m' = m \quad e' = e \quad s' = s \quad i' = i$</td>
</tr>
<tr>
<td>$E \xrightarrow{w} M$</td>
<td>$e \geq 1$</td>
<td>$m' = m + 1 \quad e' = e - 1 \quad s' = s \quad i' = i$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$M \xrightarrow{rh} M$</td>
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</tr>
</tbody>
</table>
Reachability in broadcast protocols

Typical set $I$ of initial configurations: parametrized configuration.

Typical set $D$ of final configurations: upward-closed sets.

- $U$ is an upward-closed set of configurations if
  \[ c \in U \text{ and } c' \geq c \text{ implies } c' \in U \]
  where $\geq$ is the pointwise order on $\mathbb{N}^n$.

- Example: states $M$ and $S$ of MESI protocol should be mutually exclusive
  \[ D = \{(m, e, s, i) \mid m \geq 1 \land s \geq 1\} \]

Question: Is reachability decidable if $I$ is a parametric configuration and $D$ is an upward-closed set?
First try: Forward search

Since \( I \in \mathcal{C} \) is required by condition (2), the family \( \mathcal{C} \) must contain all parametrized configurations.

Satisfies (1) - (5) but not (6). Termination fails in very simple cases.

\[(\sqcup, 0) \xrightarrow{a} (\sqcup, 1) \xrightarrow{a} (\sqcup, 2) \xrightarrow{a} \ldots\]
Second try: Backward search

Since $D \in C$ is required by condition (2), the family $C$ must contain all upward-closed sets.

**Theorem** [Abdulla et al., I&C 160, 2000], [E. et al., LICS’99]
Backward search satisfies conditions (1) - (6).

Proof in the next slides.
Proof

1. An upward-closed set can be finitely represented by
   its set of minimal elements w.r.t. the pointwise order $\leq$
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Every infinite sequence $c_1, c_2, c_3, \ldots$ of vectors of $\mathbb{N}^k$ contains a non-decreasing infinite subsequence $c_{i_1} \leq c_{i_2} \leq c_{i_3} \ldots$ (Dickson’s lemma)
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But then $m_j$ is not minimal.
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Contradiction.
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\textit{Contradiction.}
2.  $D$ is upward-closed  √
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3. If $C$ is upward-closed then so is $C \cup \text{pre}(C)$. 
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Since union of upward-closed sets is upward-closed, it suffices to prove that $\text{pre}(C)$ is upward-closed.
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   Key idea: “adding more processes to a configuration cannot disable any transition”.
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\begin{align*}
  c & \rightarrow d \in C \\
  \leq & \\
  c' & \nonumber
\end{align*}
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Pick some minimal element \( m_1 \in U_1 \).

Pick for every \( i > 1 \) some minimal element \( m_i \notin U_1 \cup \ldots \cup U_{i-1} = U_{i-1} \).

Consider the sequence \( m_1, m_2, m_3, \ldots \).
4. \( C \cap I \) is decidable. \( \surd \)

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**Contradiction to Dickson’s lemma.**
4. $C \cap I$ is decidable. √

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Contradiction to Dickson’s lemma.
Consider the sequences $C = c_1, c_2, c_3, \ldots$, where $c_i \in \mathbb{N}^k$ for all $i \geq 1$, that satisfy:

- $c_1 \leq (1, \ldots, 1)$, and
- $|c_i(j) - c_{i+1}(j)| \leq 1$ for every $i \geq 1, 1 \leq j \leq k$.

By Dickson’s lemma any such sequence contains indices $i, j$ such that $c_i \leq c_j$.

Let $J(C)$ be the smallest $j$ for which such an $i$ exist.

Let $G(k)$ be the maximum over all $C$’s of the index $J(C)$.

How fast can $G$ grow?
Consider the sequences $C = c_1, c_2, c_3, \ldots$, where $c_i \in \mathbb{N}^k$ for all $i \geq 1$, that satisfy:

- $c_1 \leq (1, \ldots, 1)$, and
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How fast can $G$ grow?

**Theorem [Mayr,Meyer, JACM ’81]**: The function $G$ is non-primitive recursive.
Consider the sequences $C = c_1, c_2, c_3, \ldots$, where $c_i \in N^k$ for all $i \geq 1$, that satisfy:

- $c_1 \leq (1, \ldots, 1)$, and
- $|c_i(j) - c_{i+1}(j)| \leq 1$ for every $i \geq 1$, $1 \leq j \leq k$.

By Dickson’s lemma any such sequence contains indices $i, j$ such that $c_i \leq c_j$.

Let $J(C)$ be the smallest $j$ for which such an $i$ exist.

Let $G(k)$ be the maximum over all $C$’s of the index $J(C)$.

**How fast can $G$ grow?**

**Theorem [Mayr,Meyer, JACM ’81]:** The function $G$ is non-primitive recursive.

Backward search may need a non-primitive recursive number of iterations.

However: Still useful in practice!
Application to the MESI-protocol

Are the states $M$ and $S$ mutually exclusive?

Check if the upward-closed set with minimal element

$$m = 1, \ e = 0, \ s = 1, \ i = 0$$

can be reached from the initial parametrized configuration

$$m = 0, \ e = 0, \ s = 0, \ i = \square$$

Proceed as follows:
Application to the MESI-protocol

Are the states $M$ and $S$ mutually exclusive?

Check if the upward-closed set with minimal element

$$m = 1, \ e = 0, \ s = 1, \ i = 0$$

can be reached from the initial parametrized configuration

$$m = 0, \ e = 0, \ s = 0, \ i = \Box$$

Proceed as follows:

$$D: \ m \geq 1 \land s \geq 1$$
Application to the MESI-protocol

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Proceed as follows:

$$D: \quad m \geq 1 \land s \geq 1$$

$$D \cup \text{pre}(D): \quad (m \geq 1 \land s \geq 1) \lor$$

$$\quad (m = 0 \land e = 1 \land s \geq 1)$$
Application to the MESI-protocol

Are the states $M$ and $S$ mutually exclusive?

Check if the upward-closed set with minimal element

$$m = 1, \ e = 0, \ s = 1, \ i = 0$$

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$$m = 0, \ e = 0, \ s = 0, \ i = \Box$$

Proceed as follows:

$$D: \ m \geq 1 \land s \geq 1$$

$$D \cup \text{pre}(D): \ (m \geq 1 \land s \geq 1) \lor (m = 0 \land e = 1 \land s \geq 1)$$

$$D \cup \text{pre}(D) \cup \text{pre}^2(D): \ D \cup \text{pre}(D)$$
Case studies

Other cache-coherence protocols: Berkeley RISC, Illinois, Xerox PARC Dragon, DEC Firefly, Futurebus +, etc.

[Delzanno, FMSD’03]:

- Model extended with more complicated guards.
- Termination guarantee gets lost.
- Upward-closed sets represented by linear constraints.
- Backward-search algorithm must be refined: Possibly more iterations, but each iteration has lower complexity.

[Emerson,Kahlon, CHARME’03,TACAS’03]:

- Restricted models still able to model the cache-coherence protocols.
- Much faster algorithms.
Symbolic search for other models

Lossy channel systems [Abdulla and Jonsson, I&C ’93], [Abdulla et al, CAV’98].

- Configuration: \( \langle q, w \rangle \), where \( q \) state and \( w = (w_1, \ldots, w_n) \) vector of words representing the current queue contents

- Family \( C \): upward-closed sets with respect to the subsequence order
  \[ abba \preceq bbaabaabababb \]
  Dickson’s lemma \( \rightarrow \) Higman’s lemma

- Backward search satisfies (1) - (6).

Timed Petri nets [Abdulla and Nylén, ICATPN’01].

- Configuration: \( \langle q, B \rangle \), where \( B \) finite bag of vectors of reals.

- Family \( C \): existential zones.
Repeated reachability in broadcast protocols

The following problem is undecidable:

Given: a broadcast protocol,
an initial parametrized configuration $p = (\sqcup, 0, \ldots, 0)$

To decide: is there an integer $n$ such that the transition system
with $(n, 0, \ldots, 0)$ as initial configuration
has an infinite computation?

Can be reformulated as a repeated reachability problem where
$I = (\sqcup, 0, \ldots, 0)$ and $D =$ set of all configurations.
Pushdown automata
Pushdown automata

Automata extended with one stack.

Transitions:

- Guards: check the topmost symbol in the stack.
- Actions: replace the topmost symbol by a fixed word.
- Notation: $\langle p, \gamma \rangle \rightarrow \langle p', v \rangle$
- Normalization: $|v| \leq 2$

We use $P$, $\Gamma$, $\Delta$ for the sets of control states, stack symbols, and rules, respectively.

Configurations: pairs $\langle p, w \rangle$, where $p$ is a control state and $w$ is a word.

(Stack, topmost symbol is the first letter.)
PDAs as models of sequential programs

Programs determined by:

- Control flow: assignments, conditionals, loops, procedure calls with parameters/return values.
- Local variables of each procedure.
- Global variables.

State space determined by:

- Program pointer.
- Values of global variables.
- Values of local variables (of current procedure).
- Activation records (return addresses, copies of locals).
Interpretation of a configuration $\langle q, \gamma \nu \rangle$:

$q$ holds values of global variables.

$\gamma$ holds (program pointer, values of local variables).

$\nu$ holds stack of (return address, saved locals).

Restriction: finite datatypes.

Correspondence between statements and rules:

$\langle q, \gamma \rangle \leftrightarrow \langle q', \gamma' \rangle$ simple statement

$\langle q, \gamma \rangle \leftrightarrow \langle q', \gamma' \gamma'' \rangle$ procedure call

$\langle q, \gamma \rangle \leftrightarrow \langle q', \epsilon \rangle$ return statement
void m() {
    if (?) {
        s(); right();
        if (?) m();
    } else {
        up(); m(); down();
    }
}

void s() {
    if (?) return;
    up(); m(); down();
}

main() {
    s();
}
Model

```c
void s() {

    var st: stack of \{s_0, \ldots, s_5, \ldots\}

    s_0: if (?) s_1: return;

    s_2: up();

    s_3: m();

    s_4: down(); s_5:

    \langle p, s_0 \rangle \leftarrow \langle p, s_2 \rangle \quad \langle p, s_0 \rangle \leftarrow \langle p, \epsilon \rangle

    \langle p, s_2 \rangle \leftarrow \langle p, up_0 s_3 \rangle

    \langle p, s_3 \rangle \leftarrow \langle p, m_0 s_4 \rangle

    \langle p, s_4 \rangle \leftarrow \langle p, down_0 s_5 \rangle \quad \langle p, s_5 \rangle \leftarrow \langle p, \epsilon \rangle

}
```
Symbolic reachability in pushdown automata

A set of configurations $C$ is regular if for every control point $p$, the set
$\{ w \in \Gamma^* \mid \langle p, w \rangle \in C \}$ is regular.

Typically, $I$ and $D$ are regular sets of configurations.
(Even very simple ones, like $\langle p, \Gamma^* \rangle$.)

Family $C$: regular sets
Backward search: Do conditions (1) - (6) hold?

1. Each regular set can be \textit{finitely} represented by a NFA. \checkmark

NFA for a pushdown system:

- \( P \) as set of initial states and \( \Gamma \) as alphabet.
- \( \langle p, \nu \rangle \) recognized if \( p \xrightarrow{\nu} q \) for some final state \( q \).

Example: \( P = \{ p_0, p_1 \} \) and \( \Gamma = \{ \gamma_0, \gamma_1 \} \)

Automaton coding the set \( \langle p_0, \gamma_0 \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle \):
2. $F \in C$  √

$\Delta = \{ \langle p_0, \gamma_0 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_1, \gamma_1 \gamma_0 \rangle \}$

$y := 0$

$x < 1$  √

$x > 1$  √

$x, y := x + ?, y + ?$  √

$y := 0$  √
2. \( F \in C \) \( \checkmark \)

3. If \( C \in C \), then \( C \cup \text{pre}(C) \in C \).
2. $F \in C \quad \checkmark$

3. If $C \in \mathcal{C}$, then $C \cup \text{pre}(C) \in \mathcal{C}$.

$$\Delta = \{ \langle p_0, \gamma_0 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_1, \gamma_1 \gamma_0 \rangle \}$$
2. $F \in \mathcal{C}$

3. If $C \in \mathcal{C}$, then $C \cup \text{pre}(C) \in \mathcal{C}$.

$$\Delta = \{ \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \hookrightarrow \langle p_1, \gamma_1\gamma_0 \rangle \}$$
2. \( F \in \mathcal{C} \)

3. If \( C \in \mathcal{C} \), then \( C \cup \text{pre}(C) \in \mathcal{C} \).

\[ \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_0, \epsilon \rangle \]
2. $F \in C$

3. If $C \in C$, then $C \cup \text{pre}(C) \in C$.

$$\langle p_0, \gamma_0 \rangle \rightarrow \langle p_0, \epsilon \rangle$$
2. $F \in C$  

3. If $C \in C$, then $C \cup \text{pre}(C) \in C$.

\[
\langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle
\]
2. \[ F \in C \]

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\langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle
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3. If $C \in \mathcal{C}$, then $C \cup \text{pre}(C) \in \mathcal{C}$.

$\langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle$
2. \( F \in C \) 

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3. If \( C \in C \), then \( C \cup \text{pre}(C) \in C \).

\[ \langle p_0, \gamma_0 \rangle \hookrightarrow \langle p_0, \epsilon \rangle \]

\[ p_0 \]

\[ p_1 \]
2. $F \in C$

3. If $C \in C$, then $C \cup \text{pre}(C) \in C.$

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\langle p_0, \gamma_0 \rangle \rightarrow \langle p_0, \epsilon \rangle
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\[ \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle \]
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\[ \checkmark \]

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\]
2. $F \in C \quad \checkmark$

3. If $C \in \mathcal{C}$, then $C \cup \text{pre}(C) \in \mathcal{C}$. \quad \checkmark

$$\Delta = \{ \langle p_0, \gamma_0 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle \}$$
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5. $C_1 = C_2$ is decidable.
6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.
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$$P = \{p_0, p_1\}, \Gamma = \{\gamma_0, \gamma_1\}$$

$$\Delta = \{\langle p_0, \gamma_0 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle\}$$
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\[ C_0 = D = \langle p_0, \gamma_0 \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle \]
6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.

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\[ \Delta = \{\langle p_0, \gamma_0 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle\} \]
\[ C_0 = D = \langle p_0, \gamma_0 \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle \]
\[ C_1 = C_0 \cup \text{pre}(C_0) = \langle p_0, (\gamma_0 + \gamma_0^2)\gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1(\epsilon + \gamma_0)\gamma_1^*(\epsilon + \gamma_0) \rangle \]
6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint.

$P = \{p_0, p_1\}, \Gamma = \{\gamma_0, \gamma_1\}$

$\Delta = \{\langle p_0, \gamma_0 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle\}$

$C_0 \quad \quad = \quad D = \langle p_0, \gamma_0 \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle$

$C_1 \quad \quad = \quad C_0 \cup \text{pre}(C_0) = \langle p_0, (\gamma_0 + \gamma_0^2) \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 (\epsilon + \gamma_0) \gamma_1^* (\epsilon + \gamma_0) \rangle$

\[\ldots\]

$C_i \quad \quad = \quad C_{i-1} \cup \text{pre}(C_{i-1}) = \langle p_0, (\gamma_0 + \ldots + \gamma_0^{i+1}) \gamma_1^* \gamma_0 \rangle \cup \langle p_1, \gamma_1 (\epsilon + \gamma_0 + \ldots + \gamma_0^i) \gamma_1^* (\epsilon + \gamma_0) \rangle$

\[\ldots\]
6. Any chain $C_1 \subseteq C_2 \subseteq C_3 \ldots$ eventually reaches a fixpoint. \hspace{1cm} FAILS!

$$P = \{p_0, p_1\}, \Gamma = \{\gamma_0, \gamma_1\}$$

$$\Delta = \{\langle p_0, \gamma_0 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \mapsto \langle p_1, \gamma_1 \gamma_0 \rangle\}$$

$$C_0 = D = \langle p_0, \gamma_0 \gamma_1 \gamma_0 \rangle \cup \langle p_1, \gamma_1 \rangle$$

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$$\langle p_1, \gamma_1 (\epsilon + \gamma_0) \gamma_1 (\epsilon + \gamma_0) \rangle$$

$$\ldots$$

$$C_i = C_{i-1} \cup \text{pre}(C_{i-1}) = \langle p_0, (\gamma_0 + \ldots + \gamma_0^{i+1}) \gamma_1 \gamma_0 \rangle \cup$$

$$\langle p_1, \gamma_1 (\epsilon + \gamma_0 + \ldots + \gamma_0^i) \gamma_1 (\epsilon + \gamma_0) \rangle$$

$$\ldots$$
However, the fixpoint

\[ \text{pre}^*(D) = \langle p_0, \gamma_0^+ \gamma_1 \gamma_0 \rangle \cup \langle p_1, \gamma_1 \gamma_0^* \gamma_1^* (\epsilon + \gamma_0) \rangle \]

is regular.

\textit{How can we compute it?}
Accelerations

By definition, \( \text{pre}(D) = \bigcup_{i \geq 0} C_i \)
where \( C_0 = D \) and \( C_{i+1} = C_i \cup \text{pre}(C_i) \) for every \( i \geq 0 \)

If convergence fails, try to compute an acceleration:
a sequence \( D_0 \subseteq D_1 \subseteq D_2 \ldots \) such that

(a) \( \forall i \geq 0: C_i \subseteq D_i \)
(b) \( \forall i \geq 0: D_i \subseteq \bigcup_{j \geq 0} C_j = \text{pre}(D) \)

Property (a) ensures capture of (at least) the whole set \( \text{pre}(D) \)
Property (b) ensures that only elements of \( \text{pre}(D) \) are captured

The acceleration guarantees termination if

(c) \( \exists i \geq 0: D_{i+1} = D_i \)
An acceleration for pushdown automata

Idea: reuse the same states
An acceleration for pushdown automata

Idea: reuse the same states

\[ \Delta = \{ \langle p_0, \gamma_0 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_0, \epsilon \rangle, \langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle \} \]
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![Diagram of pushdown automata]
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$\langle p_1, \gamma_1 \rangle \leftrightarrow \langle p_1, \gamma_1 \gamma_0 \rangle$
An acceleration for pushdown automata

Idea: reuse the same states

\( \langle p_0, \gamma_0 \rangle \mapsto \langle p_0, \epsilon \rangle \)

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An acceleration for pushdown automata

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But does it work . . .?

All predecessors are computed, and termination guaranteed.

But: we might be adding non-predecessors.
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Fortunately: correct if initial states have no incoming arcs.
The proof (1/4)

Input: Pushdown automaton \((P, \Gamma, \Delta)\), NFA \(\mathcal{A} = (Q, \Gamma, \rightarrow_0, P, F)\) recognizing a regular set \(\mathcal{C}\).

Precondition: No transition of \(\mathcal{A}\) leads to an initial state.

Output: NFA \(\mathcal{A}_{pre^*} = (Q, \Gamma, \rightarrow, P, F)\).

Postcondition: \(\mathcal{A}_{pre^*}\) recognizes \(pre^*(\mathcal{C})\).

Algorithm: Add new transitions according to the following saturation rule

If \(\langle p, \gamma \rangle \leftarrow \langle p', w \rangle\) and \(p' \xrightarrow{w} q\) in the current automaton, add a transition \((p, \gamma, q)\).
The proof (2/4)

Goal: show that $A_{pre^*}$ only recognizes words of $pre^*(C)$.

(Showing that it recognizes all words of $pre^*(C)$ is easy.)
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Notation: $\rightarrow_i$ denotes the transition relation after adding $i$ transitions to $A$. 
The proof (2/4)

Goal: show that $A_{\text{pre}^*}$ only recognizes words of $\text{pre}^*(C)$.

(Showing that it recognizes all words of $\text{pre}^*(C)$ is easy.)

Notation: $\rightarrow_i$ denotes the transition relation after adding $i$ transitions to $A$.

We show: If $p \xrightarrow[\rightarrow_i]{} w q$, then $\langle p, w \rangle \Rightarrow^* \langle p', w' \rangle$ for some $\langle p', w' \rangle$ such that $p' \xrightarrow[w']{} q$; moreover, if $q$ initial, then $w' = \epsilon$. 

The proof (2/4)

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We show: If $p \xrightarrow{w}_i q$, then $\langle p, w \rangle \Rightarrow^* \langle p', w' \rangle$ for some $\langle p', w' \rangle$ such that $p' \xrightarrow{w'}_0 q$; moreover, if $q$ initial, then $w' = \epsilon$.

Proof by induction on $i$. Basis $i = 1$ is easy.
The proof (2/4)

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$i > 1$. Let $(p_1, \gamma, q')$ be the $i$-th transition added to $A$ ($p_1$ initial state!).
The proof (2/4)

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$i > 1$. Let $\langle p_1, \gamma, q' \rangle$ be the $i$-th transition added to $\mathcal{A}$ ($p_1$ initial state!). Let $j$ be the number of times that $\langle p_1, \gamma, q' \rangle$ is used in $p \xrightarrow{w}_i q$. 
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Proof by induction on $i$. Basis $i = 1$ is easy.

$i > 1$. Let $(p_1, \gamma, q')$ be the $i$-th transition added to $A$ ($p_1$ initial state!). Let $j$ be the number of times that $(p_1, \gamma, q')$ is used in $p \xrightarrow{w}_i q$.

By induction on $j$. Basis $j = 0$ is easy.
The proof (3/4)

Step. $j > 0$. So $(p_1, \gamma, q')$ occurs in $p \xrightarrow{w} i q$. We have:

1. $p u \xrightarrow{i} q^1 p_1 \gamma \xrightarrow{i} q' v \xrightarrow{i} q$ (by ’zooming into’ $p w \xrightarrow{i} q$)
2. $\langle p_1, \gamma \rangle \hookrightarrow \langle p_2, w_2 \rangle$
3. $p_2 w_2 \xrightarrow{i} q^1 v \xrightarrow{i} q$ (by the saturation rule)
4. $\langle p, u \rangle \Rightarrow q^2 \langle p_1, \varepsilon \rangle$ (by induction hypothesis on $i$)
5. $\langle p_2, w_2 v \rangle \Rightarrow q^2 \langle p, w_2 \rangle$
6. $p' w' \xrightarrow{0} q'$ (by induction hypothesis on $j$)

Finally, if $q$ initial then $w' = \varepsilon$ because of (6) and precondition.
The proof (3/4)

Step. \( j > 0 \). So \((p_1, \gamma, q')\) occurs in \( p \xrightarrow{w_i} q \). We have:

1. \( p \xrightarrow{u_i} p_1 \xrightarrow{\gamma_i} q' \xrightarrow{v_i} q \) (by ‘zooming into’ \( p \xrightarrow{w_i} q \))
Step. $j > 0$. So $(p_1, \gamma, q')$ occurs in $p \xrightarrow{w_i} q$. We have:

1. $p \xrightarrow{u_{i-1}} p_1 \xrightarrow{\gamma_i} q' \xrightarrow{v_i} q$ (by ‘zooming into’ $p \xrightarrow{w_i} q$)

2. $\langle p_1, \gamma \rangle \hookrightarrow \langle p_2, w_2 \rangle$

3. $p_2 \xrightarrow{w_{i-1}} q' \xrightarrow{v_i} q$ (by the saturation rule)
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(3) \( p_2 \xrightarrow{w_2} q' \xrightarrow{v} q \) (by the saturation rule)

(4) \( \langle p, u \rangle \Rightarrow^* \langle p_1, \varepsilon \rangle \) (by induction hypothesis on \( i \))

(5) \( \langle p_2, w_2 v \rangle \Rightarrow^* \langle p', w' \rangle \)

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Finally, if \( q \) initial then \( w' = \varepsilon \) because of (6) and precondition.
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5. \[ \langle p_2, w_2v \rangle \Rightarrow^* \langle p', w' \rangle \]
6. \[ p' \xrightarrow{w'_0} q \] (by induction hypothesis on \( j \))

\[ \langle p, w \rangle = \langle p, u\gamma v \rangle \] (1)
Step. \( j > 0 \). So \( (p_1, \gamma, q') \) occurs in \( p \xrightarrow{w_i} q \). We have:

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(1) \hspace{2cm} (4)
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5. $\langle p_2, w_2 v \rangle \Rightarrow^* \langle p', w' \rangle$

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$$\langle p, w \rangle = \langle p, u \gamma v \rangle \Rightarrow^* \langle p_1, \gamma v \rangle \Rightarrow^* \langle p_2, w_2 v \rangle$$

(1) (4) (2)

Finally, if $q$ initial then $w' = \varepsilon$ because of (6) and precondition.
The proof (3/4)

Step. \( j > 0 \). So \((p_1, \gamma, q')\) occurs in \( p \xrightarrow{w} q \). We have:

1. \( p \xrightarrow{u} p_1 \xrightarrow{\gamma} q' \xrightarrow{v} q \) \hspace{1cm} (by ‘zooming into’ \( p \xrightarrow{w} q \))
2. \( \langle p_1, \gamma \rangle \hookrightarrow \langle p_2, w_2 \rangle \)
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6. \( p' \xrightarrow{w'} q \) \hspace{1cm} (by induction hypothesis on \( j \))

\[\langle p, w \rangle = \langle p, u \gamma v \rangle \quad \Rightarrow^* \quad \langle p_1, \gamma v \rangle \quad \Rightarrow \quad \langle p_2, w_2 v \rangle \quad \Rightarrow^* \quad \langle p', w' \rangle\]

(1) (4) (2) (5)
The proof (3/4)

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\[ \langle p, w \rangle = \langle p, u\gamma v \rangle \Rightarrow^* \langle p_1, \gamma v \rangle \Rightarrow^* \langle p_2, w_2 v \rangle \Rightarrow^* \langle p', w' \rangle \]

Finally, if $q$ initial then $w' = \epsilon$ because of (6) and precondition.

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Forward search and complexity

Symbolic forward search with regular sets can be accelerated in a similar way.

Recall input: Pushdown automaton \((P, \Gamma, \Delta)\), NFA \(A = (Q, \Gamma, \rightarrow_0, P, F)\).

Complexity of backward search: \(O(|Q|^2 \cdot |\Delta|)\) time, \(O(|Q| \cdot |\Delta| + |\rightarrow_0|)\) space.

Complexity of forward search: \(O(|P| \cdot |\Delta| \cdot (|Q \setminus P| + |\Delta|) + |P| \cdot |\rightarrow_0|)\) time and space.
Reachable configurations of the plotter program
Repeated reachability for pushdown systems

Let $I = \langle p_0, \gamma_0 \rangle$ and $D = \langle p, \Gamma^* \rangle$.

$D$ can be repeatedly reached from $I$ iff

\[
\langle p_0, \gamma_0 \rangle \xrightarrow{*} \langle p', \gamma w \rangle \\
\text{and} \\
\langle p', \gamma \rangle \xrightarrow{*} \langle p, v \rangle \xrightarrow{*} \langle p', \gamma u \rangle
\]

for some $p', \gamma, w, v, u$.

Repeated reachability can be reduced to computing several $\text{pre}^*$. 
To know more

Pushdown automata usually called pushdown processes in our context.

They are equivalent to recursive state machines.

The class of one-state PDAs is interesting, usually studied under the name Basic Process Algebra (BPA) or context-free processes.


Tools: Moped, available online at http://www.informatik.uni-stuttgart.de/fmi/szs/tools/moped/

Technology transfer: the Static Driver Verifier (Microsoft) see http://www.microsoft.com/whdc/devtools/tools/SDV.mspx
(Lossy) Channel Systems
(Lossy) Channel systems

Automata extended with **channels** (unbounded queues)

**Send** transitions: no guard, action sends message to the channel.

**Receive** transitions: guard checks if the channel is nonempty, action removes the first message.

**Loss** transitions: self-loops, no guard, action removes an arbitrary message.
(Lossy) Channel systems

Automata extended with channels (unbounded queues)

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Receive transitions: guard checks if the channel is nonempty, action removes the first message.

Loss transitions: self-loops, no guard, action removes an arbitrary message.
Case study: A sliding window protocol
Symbolic reachability for (lossy) channel systems

Perfect channels: Turing powerful model, even with only one channel.

Lossy channels:

- Backward search: decidable for $D$ upward-closed set
- Forward search: Choose $C$ as the set of simple regular expressions (SREs).

  Atomic expression: $(a + \epsilon) | (a_1 + \ldots + a_m)^*$
  Product: $e_1 e_2 \ldots e_n$
  SRE: $p_1 + \ldots + p_n$

  SREs satisfy conditions (1)-(5) (exercise), but not (6).
  The fixpoint is an SRE, but it cannot be effectively computed (!), and so no ‘perfect’ acceleration can exist.
Compute a **symbolic reachability graph** with elements of $C$ as nodes:

- Add $I$ as first node
- For each node $C$ and each transition $t$, add an edge $C \xrightarrow{t} \text{post}[t](C)$
Acceleration through loops

Compute a symbolic reachability graph with elements of $C$ as nodes:

- Add $I$ as first node
- For each node $C$ and each transition $t$, add an edge $C \xrightarrow{t} \text{post}[t](C)$

Replace $C \xrightarrow{\sigma} \text{post}[\sigma](C)$ by $C \xrightarrow{\sigma} X$, where $X$ satisfies

- $\text{post}[\sigma](C) \subseteq X$, and
- $X$ contains only reachable configurations.
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A loop is a sequence of transitions leading from a control state to itself.

Acceleration: given a loop $C \xrightarrow{\sigma} \text{post}[\sigma](C)$, replace $\text{post}[\sigma](C)$ by

$$X = \text{post}[\sigma^*](C) = C \cup \text{post}[\sigma](C) \cup \text{post}[\sigma^2](C) \cup \ldots$$
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$$X = \text{post}[\sigma^*](C) = C \cup \text{post}[\sigma](C) \cup \text{post}[\sigma^2](C) \cup \ldots$$

Question: find a suitable class of loops such that $\text{post}[\sigma^*](C)$ belongs to $C$. 
An acceleration for lossy channel systems

**Theorem [Abdulla, Bouajjani, Jonsson, CAV’98]:** For any loop $\sigma$ of a lossy channel system and any SRE $r$, the set $\text{post}[\sigma^*](r)$ is an SRE that can be computed in quadratic time in the size of $r$. 
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Use in verification:
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Use in verification:

Preselect a set of loops (e.g., those corresponding to simple cycles).
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Given a set of configurations, compute first the effect of executing each of the loops infinitely often, and then compute for each transition the effect of computing it.
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Pray for termination.
Channel contents of the sliding window protocol

<table>
<thead>
<tr>
<th>States</th>
<th>Mess. channel</th>
<th>Ack. channel</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1, r_1$</td>
<td>$(m_2 + m_3)^* (m_1 + m_3)^* (m_1 + m_2)^*$</td>
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</tr>
<tr>
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The learning approach

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Recent alternative [*Vardhan, Sen, Viswanathan, Agha, FSTTCS ’04*]: apply *learning algorithms for regular languages*. 
Angluin’s learning setting [I&C ’87]

Two agents, the Teacher and the Learner.

The Teacher knows a regular language $L \subseteq \Sigma^*$.

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- **Membership queries**: The Learner produces $w \in \Sigma^*$, and asks if $w \in L$. The Teacher answers yes/no.
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**Question**: give an algorithm (a strategy) for the Learner.
Structure of Angluin’s algorithm

The Learner repeatedly asks membership queries until it has enough information to state a hypothesis.
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Completeness: the Learner eventually produces $L$ as hypothesis.

Complexity: polynomial in the size of the minimal DFA for $L$. 

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Define an execution as a pair \((\sigma, c)\) where \(c\) is a configuration and \(\sigma\) is a witness, i.e., a sequence of transitions that can be executed from some initial configuration and whose execution leads to \(c\).
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… but equivalence queries still hopeless.
Define a marked transition sequence (MTS) as a pair \((\sigma, c)\), where \(\sigma\) is a sequence of transition names and \(c\) is a configuration. Notice that executions are MTS.
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Adapt Angluin’s algorithm to learn either

- (DE) a dangerous execution, or
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Replace equivalence queries by containment queries.
Containment queries

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2. $H \not\supseteq \text{Exec}$ and $H \cap D \neq \emptyset$, then the Teacher returns $(\sigma, c) \in H \cap D$. The Learner checks whether $(\sigma, c) \in \text{Exec}$:
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   2.2. if $(\sigma, c) \not\in \text{Exec}$, then $(\sigma, c) \in H \oplus \text{Exec}$, and so the Learner has got a counterexample.

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The clever idea:

If $c \xrightarrow{\tau} c'$, then say $(\sigma, c) \rightarrow (\sigma t, c')$. Given a set $M$ of MTSs, let

$$\text{post}(M) = \{ m \mid \exists m' \in M \land m' \rightarrow m \}$$

$\text{Exec}$ is the least fixed point of the equation $X = \mathcal{F}(X)$ where

$$\mathcal{F}(X) =_{\text{def}} \{ (\epsilon, c) \mid c \in I \} \cup \text{post}(X)$$

By standard fixed point theory: if $\mathcal{F}(H) \subseteq H$, then $H \supseteq \text{Exec}$.
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We replace the query $H \supseteq \text{Exec}$ by the query $\mathcal{F}(H) \subseteq H$. 
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1. $\neg \mathcal{F}(H) \subseteq H$, then $H \supseteq \text{Exec}$, and we can proceed as before.

2. $\neg \mathcal{F}(H) \setminus H \neq \emptyset$, then the Teacher chooses $m \in \mathcal{F}(H) \setminus H$.

So we have $m \in \{(\epsilon, c) | c \in I \} \cup \text{post}(H)$ and $m \notin H$.

2.1 If $m \in \{(\epsilon, c) | c \in I \}$, then $m \in \text{Exec} \setminus H$.

The Teacher returns $m$ as counterexample.

2.2 If $m \in \text{post}(H)$, the Teacher computes $m' \in H$ with $m' \rightarrow m$.

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2.2.2 If $m' \in \text{Exec}$, then $m \in \text{Exec}(m' \rightarrow m)$ holds and so $m \in \text{Exec} \setminus H$.

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Remaining problems:

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**Theorem (exercise):** If \( M \) is a regular set of MTSs of a (lossy) channel system, then so is \( \text{post}(M) \). Moreover, \( \text{post}(M) \) can be effectively computed.
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**Corollary:** If $I$ is a regular set of configurations and $H$ is a regular hypothesis of a (lossy) channel system, then $\mathcal{F}(H)$ is also regular and can be effectively computed.
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Algorithms for the remaining problems follow easily from the Corollary.
Some observations

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In practice, the assumption ‘$Exec$ is regular’ is stronger than the assumption ‘$post^* (I)$ is regular’. For instance, $post^* (I)$ is always regular for a pushdown system (assuming $I$ regular), while $Exec$ is context-free.
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The assumption ‘$Exec$ is regular’ may depend on the encoding use to represent a pair $(\sigma, c)$ as a word.