Beyond Big-Oh analysis in automata theory

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A bit of satire . . .

Theoretical computer scientists as classifiers.

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- A paper deserves publishing iff it provides new or better bounds.

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- Automata theory for verification very much profits from "beyond Big-Oh" analysis and prototype implementations.

Today's menu

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Universality of finite automata

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Complexity:

$O(2^{|A|})$ time and space, and $\Theta(2^{|A|})$ for some family.

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If the powerset construction generates states $Q_1 \subseteq Q_2$, redirect Q_2's incoming arcs to Q_1 and remove Q_2.

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- Assume $Q_1 \subseteq Q_2$. We have $L_B(Q_1) \subseteq L_B(Q_2)$ and if B universal then $L_B(Q_1) = L_B(Q_2)$.

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- Let B' be the result of the operation. Then $L_{B'} \subseteq L_B$ and if B universal then $L_{B'} = L_B$.

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- Let B' be the result of the operation. Then $L_{B'} \subseteq L_B$ and if B universal then $L_{B'} = L_B$.
- So B' universal iff B universal iff A universal.

Potential application to verification

Typical scenario

- System: communicating automata $A_1, A_2, \ldots, A_n$.
- Specification (allowed behaviour): automaton B.
- System's behaviour: automaton $A = A_1 \otimes A_2 \otimes \ldots \otimes A_n$.
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- Check emptiness of $A \times \overline{B}$.

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- Compute $A = A_1 \otimes \ldots \otimes A_n$. Possible blowup!
- Check emptiness of $A \times \overline{B}$.

Alternative approach: $L(A) \subseteq L(B)$ iff $\overline{L(A)} \cup L(B) = \Sigma^*$

- Compute $\overline{A} = \overline{A}_1 \oplus \ldots \oplus \overline{A}_n$.
- Check universality of $A + \overline{B}$. Possible blowup!

Emptiness of Büchi automata

The problem

Given: a Büchi automaton A.
Decide: is $L(A) = \emptyset$?

Lassos

A is nonempty iff it contains an accepting lasso: a path leading from some initial state to some accepting state, followed by a cycle.

A trivial quadratic algorithm

The algorithm

(1) Compute all reachable final states.

(2) For every final state q:
 - check if q is reachable from itself.
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- (1) takes $O(|A|)$ time.
- (2) takes $O(|A|^2)$ time, and there is a family of graphs for which it takes $\Theta(|A|^2)$.

A first linear algorithm: double-DFS [CVWY91]

(1) Use DFS to compute a list $\alpha_1, \alpha_2, \ldots, \alpha_k$ of all reachable accepting states

(2) For $i = 1$ to k:
- use DFS to check if α_i is reachable from itself
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Answer "empty".

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(1) Use DFS to compute a list $\alpha_1, \alpha_2, \ldots, \alpha_k$ of all reachable accepting states sorted in postorder.
(a state is added to list when **backtracking** from it)

(2) For $i = 1$ to k:
- use a modified DFS to check if α_i is reachable from itself without visiting any state reachable from $\alpha_1, \ldots, \alpha_{i-1}$.
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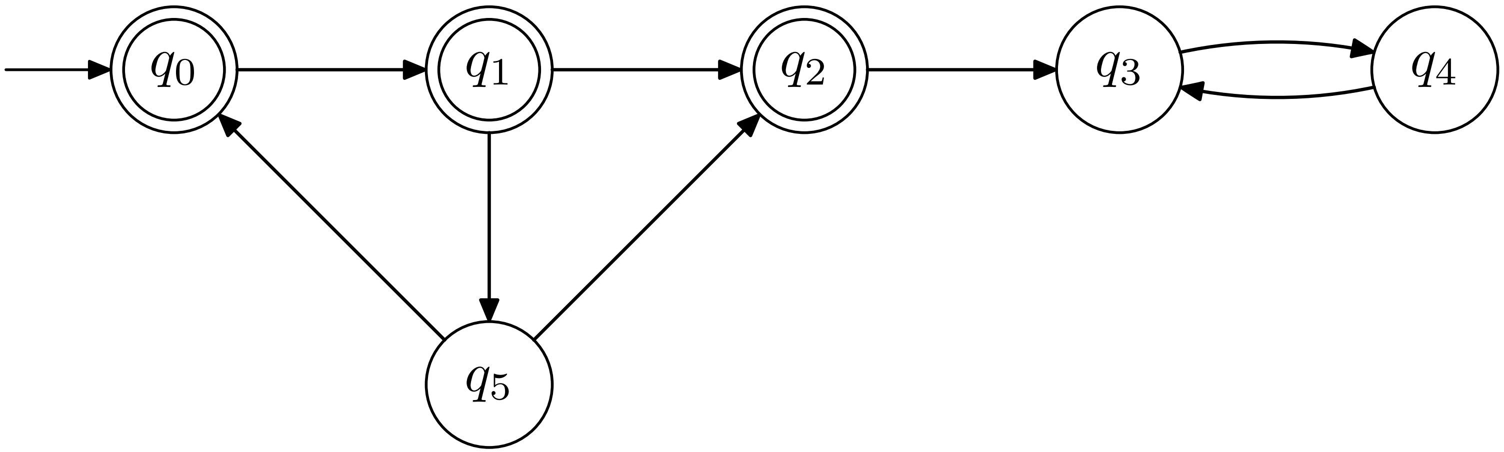

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- For each state we have three possible situations:
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  - discovered by both phases.
- 2 additional bits per (reachable) state.

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- Potential problem: some cycle contains α_2, some transition of the cycle is reachable from α_1.
- Call these cycles blocked.

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- If there is a blocked cycle, then $\alpha_1 \rightsquigarrow \alpha_2$.
- If $(\alpha_1 \rightsquigarrow \alpha_2 \wedge \alpha_2 \rightsquigarrow \alpha_1)$ then some cycle contains α_1.
- So it suffices to guarantee: if $\alpha_1 \rightsquigarrow \alpha_2$ then $\alpha_2 \rightsquigarrow \alpha_1$.
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- Look at DFS as a recursive procedure *dfs*(q).
- Let *ca*(q) denote the time at which *dfs*(q) is called.
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- Let *ca*(q) denote the time at which *dfs*(q) is called.
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  (The search backtracks from q.)
- Postorder assumption: *ret*$(\alpha_1) <$ *ret*(α_2).

Lemma

Assume $\mathit{ret}(\alpha_1) < \mathit{ret}(\alpha_2)$. If $\alpha_1 \rightsquigarrow \alpha_2$ then $\alpha_2 \rightsquigarrow \alpha_1$.

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Proof:

- By proper nesting of calls we have either:
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- Case 1: $ca(\alpha_1) < ret(\alpha_1) < ca(\alpha_2) < ret(\alpha_2)$.
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- Case 2: $ca(\alpha_2) < ca(\alpha_1) < ret(\alpha_1) < ret(\alpha_2)$.
  Then $\alpha_2 \rightsquigarrow \alpha_1$.

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  Double-DFS requires to store paths for all accepting states.

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- Counterexample: just pop the call stack!
- Correctness: Easy. The second searches are exactly as in the double-DFS algorithm.

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Definition

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The nested-DFS algorithm is not optimal!

Minor improvements

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If the second search finds a state that is currently in the call stack of the first search, answer “nonempty”.

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[Schwoon, E. 05]

These two improvements still require only 2 additional bits per state: four-colour algorithm.

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Question

Are there optimal (linear-time) algorithms?

SCC-based algorithms

Approach

- Identify the reachable (nontrivial) SCCs of A.
- Check if some of them contains an accepting state.

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- Root of a SCC: the first node of the SCC discovered by the DFS.
  (The definition of root refers to a particular, fixed DFS-run!)
- If ρ is a root, then at time $ret(\rho)$ the DFS has discovered all nodes of ρ's SCC and its descendants in the dag.

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Basic notions

- Automaton $A \Rightarrow$ dag of SCCs.
- Root of a SCC: the first node of the SCC discovered by the DFS.
  (The definition of root refers to a particular, fixed DFS-run!)
- If ρ is a root, then at time $ret(\rho)$ the DFS has discovered all nodes of ρ's SCC and its descendants in the dag.

First idea

- Push all discovered nodes in a new stack (Tarjan's stack).
- For every root ρ: at time $ret(\rho)$, pop from Tarjan's stack until ρ is popped; the popped nodes constitute ρ's SCC.

Tarjan and GOD's algorithm

GOD's contribution: Oracle

For a given state q oracle decides if q is a root.

T(q)
    push(q, $Stack$);
    for each transition $q \to r$
        if r not yet explored then T(r)
    if q is a root then
        repeat $s := pop(Stack)$ until $s = q$

Implementing the oracle

Problem

The algorithm must identify the roots of the SCCs.
But the SCCs are what we want to compute!

Implementing the oracle

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But the SCCs are what we want to compute!

Second idea

- Annotate each state q with $ca(q)$ and a lowlink-number $lowlink(q)$.
  (Order induced by call numbers is all that matters)

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But the SCCs are what we want to compute!

Second idea

- Annotate each state q with $ca(q)$ and a lowlink-number $lowlink(q)$.
  (Order induced by call numbers is all that matters)
- $lowlink(q)$: lowest $ca(r)$ of states r satisfying
  - q and r lie in the same SCC, and
  - r reachable from q through states not yet discovered at time $ca(q)$.

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  - q and r lie in the same SCC, and
  - r reachable from q through states not yet discovered at time $ca(q)$.
- $lowlink(q) \leq ca(q)$ for every state q.

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The algorithm must identify the roots of the SCCs.
But the SCCs are what we want to compute!

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- $lowlink(q)$: lowest $ca(r)$ of states r satisfying
  - q and r lie in the same SCC, and
  - r reachable from q through states not yet discovered at time $ca(q)$.
- $lowlink(q) \leq ca(q)$ for every state q.
- Fact: $lowlink(q) = ca(q)$ if and only if q is a root.

Tarjan's algorithm

Miracle

lowlink(q) can be easily determined at time *ret*(q).

Tarjan's algorithm

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```
T(q)
    push(q, Stack);
    for each transition q → r
        if r not yet explored then
            T(r);
            r.lowlink := min(q.lowlink, r.lowlink)
        else if r ∈ Stack then
            r.lowlink := min(q.lowlink, r.ca)
    if q.lowlink = q.ca then
        repeat s := pop(Stack) until s = q
```

Geldenhuys and Valmari's algorithm [GV04]

- A direct modification of Tarjan's algorithm for emptiness checking is non-optimal.

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- A direct modification of Tarjan's algorithm for emptiness checking is non-optimal.
- Requires to completely explore an SCC before it is popped from the stack.

Main observation of [GV04]:

α belongs to a cycle iff $T(\alpha)$ reaches some state r satisfying two conditions:

- $r \in \mathit{Stack}$, and
- $\mathit{lowlink}(r) < \mathit{ca}(\alpha)$.

Geldenhuys and Valmari's algorithm [GV04]

Add a new parameter to the procedure to keep track of the last visited accepting state.

```
GV(q, α)
    push(q, Stack);
    for each transition q → r
        if r not yet explored then
            if r accepting then GV(r, r) else GV(r, α);
            r.lowlink := min(q.lowlink, r.lowlink)
        else if r ∈ Stack then
            if r.lowlink < α.ca then report "nonempty";
            r.lowlink := min(q.lowlink, r.ca)
    if q.lowlink = q.ca then
        repeat s := pop(Stack) until s = q
```

End of the story? No!

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Time complexity

[GV04] requires only one post op per state.

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- [GV04] requires to store two numbers per state plus a third number for Tarjan's stack ($3 \cdot \log n$ bits per state).

End of the story? No!

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[GV04] requires only one post op per state.

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- [GV04] requires to store two numbers per state plus a third number for Tarjan's stack ($3 \cdot \log n$ bits per state).
- Compare with 2 bits per state of nested-DFS or the four-colour algorithm.

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Generalized Büchi automata

- LTL $\rightarrow$ Büchi translations yield generalized BA.

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- LTL $\rightarrow$ Büchi translations yield generalized BA.
- GBA with n states and k acceptings sets $\rightarrow$ BA with $n \cdot k$ states. Expensive!

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Generalized Büchi automata

- LTL $\rightarrow$ Büchi translations yield generalized BA.
- GBA with n states and k acceptings sets $\rightarrow$ BA with $n \cdot k$ states. Expensive!
- Neither nested-DFS nor GV can be extended to GBA.

Question

Do optimal algorithms exist that

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- require less memory, and
- can be easily extended to GBAs?

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First idea

Partition Stack into Roots and Nonroots, keeping the following invariant:

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Partition Stack into Roots and Nonroots, keeping the following invariant:

- Roots contains all nodes that are roots of the part of the graph explored so far .
- Nonroots: contains all nodes that are non-roots of the part of the graph explored so far .

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- Key insight: q is a root iff it is a root of the part of the graph explored at time $ret(q)$.

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- So we can check if q is a root by checking $q = top(Roots)$ at time $ret(q)$.

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Partition Stack into Roots and Nonroots, keeping the following invariant:

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- Key insight: q is a root iff it is a root of the part of the graph explored at time $ret(q)$.
- So we can check if q is a root by checking $q = top(Roots)$ at time $ret(q)$.
- New problem: to keep the invariant.

Couvreur, Gabow, and GOD's algorithm

GOD's contribution: oracle to keep the invariant

- For $q \rightarrow r$, the oracle decides if q reachable from r: $r \rightsquigarrow q$.

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- For $q \rightarrow r$, the oracle decides if q reachable from r: $r \rightsquigarrow q$.

Couvreur, Gabow, and GOD's algorithm

GOD's contribution: oracle to keep the invariant

- For $q \longrightarrow r$, the oracle decides if q reachable from r: $r \rightsquigarrow q$.
- Observe: if $r \rightsquigarrow q$ then r belongs to a cycle.

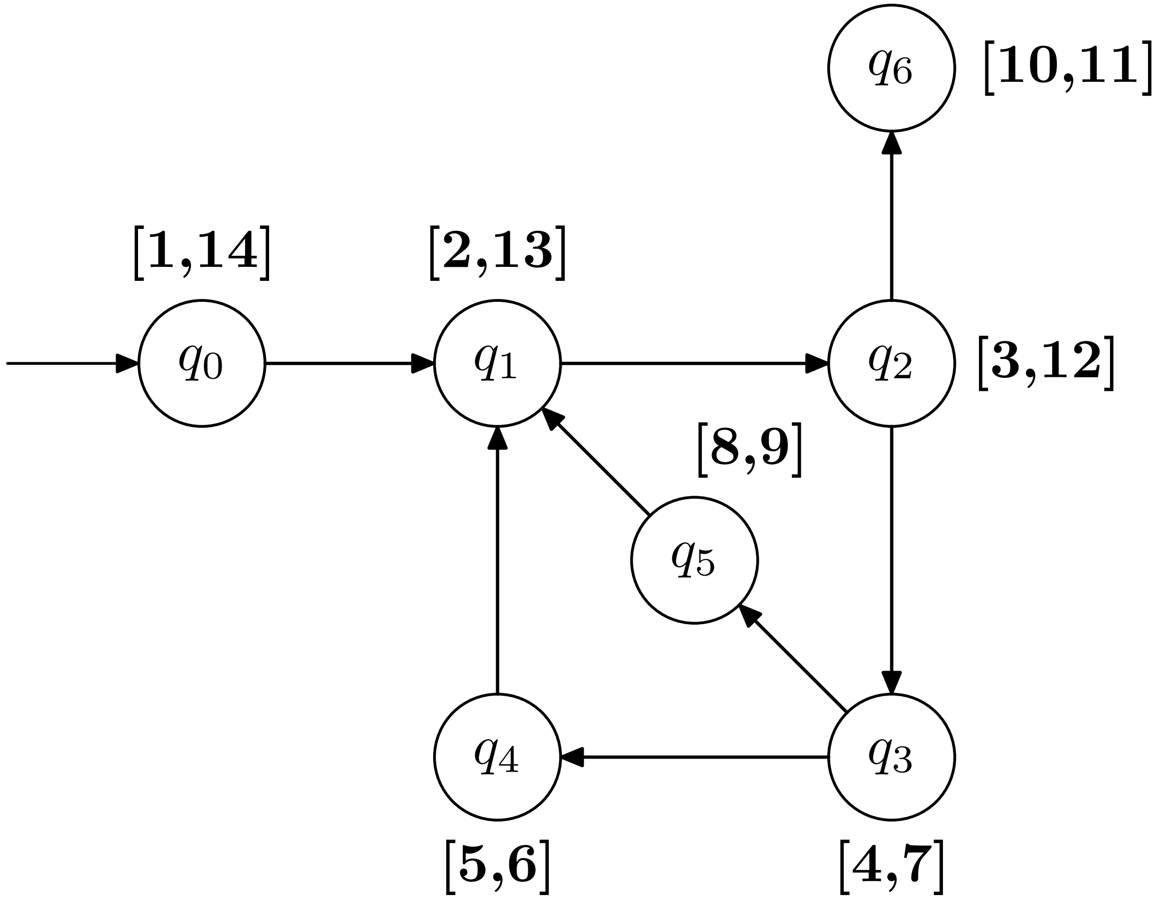
Couvreur, Gabow, and GOD's algorithm

GOD's contribution: oracle to keep the invariant

- For $q \rightarrow r$, the oracle decides if q reachable from r: $r \rightsquigarrow q$.
- Observe: if $r \rightsquigarrow q$ then r belongs to a cycle.
- We show: no node in Roots discovered after r can be a root.

```
GCG(q)
    push(q, Roots);
    for each transition q → r
        if r not yet explored then GCG(r)
        elseif r ⇝ q then
            repeat
                s :=pop(Roots); push(Nonroots);
                if s is accepting report "nonempty"
            until ca(s) ≤ ca(r);
            push(s, Roots); pop(Nonroots)
    if top(Roots) = q then
        pop(Roots);
        while ca(top(Nonroots)) > ca(q)
            pop(Nonroots)
```

Example



Time	Stack content
5	$q_4\, q_3\, q_2\, q_1\, q_0$
6	$q_1\, q_0$
8	$q_5\, q_1\, q_0$
9	$q_1\, q_0$
10	$q_6\, q_1\, q_0$
12	$q_1\, q_0$
14	ϵ

Correctness and optimality

Correctness I

If s is popped at line 7, then it belongs to a cycle containing r.

Correctness and optimality

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If s is popped at line 7, then it belongs to a cycle containing r.

Proof:

- Situation: $q \rightarrow r \rightsquigarrow q$, $s \in \mathit{Roots}$, $\mathit{ca}(s) > \mathit{ca}(r)$.

Correctness and optimality

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If s is popped at line 7, then it belongs to a cycle containing r.

Proof:

- Situation: $q \to r \rightsquigarrow q$, $s \in Roots$, $ca(s) > ca(r)$.
- We show $\rho_r \rightsquigarrow s \rightsquigarrow q \to r \rightsquigarrow \rho_r$.

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- s is a DFS-ascendant of q, and so $s \rightsquigarrow q$.

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  Because $s \in Roots$, and $Roots$ subset of DFS-stack.

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- s is a DFS-ascendant of q, and so $s \rightsquigarrow q$.
  Because $s \in Roots$, and $Roots$ subset of DFS-stack.
- ρ_r is a DFS-ascendant of s, and so $\rho_r \rightsquigarrow s$.
  Since $q \rightarrow r \rightsquigarrow q$, we have $\rho_r = \rho_q$, and so ρ_r is a DFS-ascendant of q.

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- s is a DFS-ascendant of q, and so $s \rightsquigarrow q$.
  Because $s \in Roots$, and $Roots$ subset of DFS-stack.
- ρ_r is a DFS-ascendant of s, and so $\rho_r \rightsquigarrow s$.
  Since $q \to r \rightsquigarrow q$, we have $\rho_r = \rho_q$, and so ρ_r is a DFS-ascendant of q.
  So either ρ_r is DFS-ascendant of s or vice versa.

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If s is popped at line 7, then it belongs to a cycle containing r.

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- We show $\rho_r \rightsquigarrow s \rightsquigarrow q \rightarrow r \rightsquigarrow \rho_r$.
- s is a DFS-ascendant of q, and so $s \rightsquigarrow q$.
  Because $s \in Roots$, and $Roots$ subset of DFS-stack.
- ρ_r is a DFS-ascendant of s, and so $\rho_r \rightsquigarrow s$.
  Since $q \rightarrow r \rightsquigarrow q$, we have $\rho_r = \rho_q$, and so ρ_r is a DFS-ascendant of q.
  So either ρ_r is DFS-ascendant of s or vice versa.
  But s cannot be a DFS-ascendant of ρ_r because $ca(\rho_r) \leq ca(r) < ca(s)$.

Correctness and optimality

Correctness II

If a state s is popped at line 7 and $ca(s) > ca(r)$, then it is not a root.

Correctness and optimality

Correctness II

If a state s is popped at line 7 and $ca(s) > ca(r)$, then it is not a root.

Proof:

- s belongs to a cycle containing r, and, since $ca(s) > ca(r)$, it is not a root.

Correctness and optimality

Correctness III + Optimality

Every reachable state q belonging to some cycle is eventually popped at line 7.
Moreover, q is popped immediately after any cycle containing it is completely explored.

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Proof:

- Fix a cycle C containing q.
- Let r be the last successor of q along C such that at time $ca(q)$ there is a path of unexplored states from q to r (count q as unexplored, possibly $q = r$).

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- Let r be the last successor of q along C such that at time $ca(q)$ there is a path of unexplored states from q to r (count q as unexplored, possibly $q = r$).
- Let s be the successor of r along C.

Correctness and optimality

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- Let r be the last successor of q along C such that at time $ca(q)$ there is a path of unexplored states from q to r (count q as unexplored, possibly $q = r$).
- Let s be the successor of r along C.
- $ca(s) \leq ca(q) \leq ca(r)$, and so $ca(s) \leq ca(r)$.

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Moreover, q is popped immediately after any cycle containing it is completely explored.

Proof:

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- Let r be the last successor of q along C such that at time $ca(q)$ there is a path of unexplored states from q to r (count q as unexplored, possibly $q = r$).
- Let s be the successor of r along C.
- $ca(s) \leq ca(q) \leq ca(r)$, and so $ca(s) \leq ca(r)$.
- So q is popped at line 7 when $q \rightarrow r$ is explored, or earlier.

Correctness and optimality

Correctness III

Every state discovered by the search and not belonging to any cycle is eventually popped at line 12.

Proof:

Easy.

Implementing the oracle

Lemma

Asume the oracle is asked at time t whether $r \rightsquigarrow q$.
The answer is "yes" iff $t < ret(\rho_r)$.

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Asume the oracle is asked at time t whether $r \rightsquigarrow q$.
The answer is "yes" iff $t < ret(\rho_r)$.

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Proof:

- Situation: $ca(q) \leq t < ret(q)$, $q \rightarrow r$, $ca(r) \leq t$.
- Assume $r \rightsquigarrow q$. If $t \geq ret(\rho_r)$, then $t \geq ret(q)$, contradiction. So $t < ret(\rho_r)$

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- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.

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- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.
  By postorder lemma, $ret(\rho_r) < ret(q)$.

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- Assume $r \rightsquigarrow q$. If $t \geq ret(\rho_r)$, then $t \geq ret(q)$, contradiction. So $t < ret(\rho_r)$
- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.
  By postorder lemma, $ret(\rho_r) < ret(q)$.
  Case 1: $ca(\rho_r) < ret(\rho_r) < ca(q) \leq t < ret(q)$. Done.

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Asume the oracle is asked at time t whether $r \rightsquigarrow q$.
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- Situation: $ca(q) \leq t < ret(q)$, $q \rightarrow r$, $ca(r) \leq t$.
- Assume $r \rightsquigarrow q$. If $t \geq ret(\rho_r)$, then $t \geq ret(q)$, contradiction. So $t < ret(\rho_r)$
- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.
  By postorder lemma, $ret(\rho_r) < ret(q)$.
  Case 1: $ca(\rho_r) < ret(\rho_r) < ca(q) \leq t < ret(q)$. Done.
  Case 2: $ca(q) < ca(\rho_r) \leq ca(r) < ret(\rho_r) < ret(q)$.
  Since at time t we are executing $dfs(q)$, we have $ret(\rho_r) < t \leq ret(q)$.

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The answer is "yes" iff $t < ret(\rho_r)$.

Idea

- Recall $ca(r) \leq t$.
- At time $ret(\rho)$ removes all nodes from ρ's SCC from Rots and Nonroots.
- So r stays in Stack exactly during the interval $[ca(r), ret(root(t))]$, and therefore:

Implementing the oracle

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Idea

- Recall $ca(r) \leq t$.
- At time $ret(\rho)$ removes all nodes from ρ's SCC from Rots and Nonroots.
- So r stays in Stack exactly during the interval $[ca(r), ret(root(t))]$, and therefore:

  $t < ret(\rho_r)$ iff $r \in Roots \cup Nonroots$ at time t.

Couvrer and Gabow's algorithm [C99,G00]

```
GCG(q)
    push(q, Roots);
    for each transition q → r
        if r not yet explored then GCG(r)
        elseif r ∈ Roots ∪ Nonroots then
            repeat
                s :=pop(Roots); push(Nonroots);
                if s is accepting report "nonempty"
            until ca(s) ≤ ca(r);
            push(s, Roots); pop(Nonroots)
    if top(Roots) = q then
        pop(Roots);
        while ca(top(Nonroots)) > ca(q)
            pop(Nonroots)
```

Extension to generalized Büchi automata

Store for each state $q \in Roots$ a subset $q.acc$ of accepting sets, maintaining the following invariant:

- $q.acc$ contains all the accepting sets intersecting q's SCC in the part of the graph explored so far.

Extension to generalized Büchi automata

Store for each state $q \in Roots$ a subset $q.acc$ of accepting sets, maintaining the following invariant:

- $q.acc$ contains all the accepting sets intersecting q's SCC in the part of the graph explored so far.

When GC(q) pops a cycle, add all the acc's of the popped states to $q.acc$.

```
EGC(q)
    push(q, Roots);
    q.acc := accepting sets containing q;
    for each transition q → r
        if r not yet explored then EGC(r)
        elseif r ∈ Roots ∪ Nonroots then
            repeat
                s :=pop(Roots); push(s, Nonroots);
                q.acc := q.acc ∪ s.acc
            until ca(s) ≤ ca(r);
            push(s, Roots); pop(Nonroots);
            if q.acc = all accepting sets report "nonempty"
    if q = top(Roots) then
        pop(Roots);
        while ca(top(Nonroots)) > ca(q)
            pop(Nonroots)
```

Couvreur's observation [C99]

The SCC of a root can also be determined as follows:

- Introduce one extra bit b_q for evey state q. Initially $b_q = 0$.
- For every root ρ: at time $ret(\rho)$ conduct a DFS to set to 1 the bits of all states reachable from ρ.
- The set of states that had to be flipped constitute ρ's SCC.

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- For every root ρ: at time $ret(\rho)$ conduct a DFS to set to 1 the bits of all states reachable from ρ.
- The set of states that had to be flipped constitute ρ's SCC.

Gets rid of Nonroots, but requires one extra DFS.

End of the story? No!

Černá and Pelánek's observation [ČP03]

- Many LTL specifications are translated into weak Büchi automata.
- The four-colour algorithm without the second search is correct for weak automata.

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Schwoon and E. [SE05]

The four-colour algorithm without the second searches is optimal for weak automata.

End of the story?

	Nested-DFS	SCC-based
Time	2 post ops	1/2 post op
Space	2 bits	2/1 numbers
Optimal	Only for WBA	Yes
Ext. to GBA	No	Yes

End of the story?

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Time	2 post ops	1/2 post op
Space	2 bits	2/1 numbers
Optimal	Only for WBA	Yes
Ext. to GBA	No	Yes

Practical relevance of differences in space complexity

- Small when state descriptors explicitely stored.
  (state descriptors are often dozens of bytes long)
- Large when state-hashing is applied.
  (one or two bits for storing a state)

Open questions

- Are there optimal algorithms requiring only a constant number of additional bits per state?

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- Can a shortest counterexample be computed in linear time?

Universal search

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- We give a universal algorithm that is optimal for every F.
- Corollary: if constants don't matter we are all useless!

A bit more formally . . .

- Fix a formal system (i.e., ZF).
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Theorem (Levin)

There is an algorithm $U[F, x]$ such that $U[F, -]$ is optimal for every provably computable function F.

A non-optimal algorithm $U_1[F, -]$

We describe first an obviously correct algorithm $U_1[F, -]$.
On input x, $U_1[F, -]$ behaves as follows:

- $U_1[F, -]$ enumerates all pairs $\Pi = (P, D)$, where P program and D derivation of the formal system.
  Let $\Pi_1, \Pi_2, \Pi_3 \ldots$ be this enumeration.
- For every $\Pi_i = (P_i, D_i)$: $U_1[F, -]$ checks if D_i is a proof that P_i computes F. If so, $U_1[F, -]$ computes $P_i[x]$ and stops.

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The algorithm $U[F, -]$

$U[F, x]$ dovetails the computations of $U_1[F, -]$. It spends:

- every second step on Π_1;
- every second step of the remaining ones on Π_2;
- every second step of the remaining ones on Π_3, etc.

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If P runs in $f(n)$ time, then $U[F, -]$ runs in $O(f(n))$ time.

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  $f(x) + 2f(x) + 2^2 f(x) + \ldots + 2^i f(x) = (2^{i+1} - 1) f(x)$

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- Steps spent on $\Pi_{i+1}, \Pi_{i+2}, \ldots$:
  $\frac{1}{2}f(x) + \frac{1}{4}f(x) + \ldots + 1 \leq f(x) = f(x)$

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- It is not only about heuristics and hacking: good theory is waiting for us there.