Beyond Big-Oh analysis in automata theory

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A bit of satire . . .

Theoretical computer scientists as classifiers.
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Theoretical computer scientists as classifiers.

Definition

A theoretical computer scientist (TCS) is a (possibly non-terminating) algorithm that gets a problem $P$ as input and outputs a lower bound $\Omega(LB)$ and an upper bound $O(UB)$. A TCS is sober if $LB \leq UB$, otherwise is drunk. A TCS is good iff it writes papers that deserve publishing. A paper deserves publishing iff it provides new or better bounds.
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However, implementations are sometimes needed to please reviewers and research councils. Fortunately, they can be left to another class of human beings:
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Implementing algorithms is a mechanical task. It brings a theoretician neither new insights nor “scientific glory”.

However, implementations are sometimes needed to please reviewers and research councils. Fortunately, they can be left to another class of human beings: students.
Theoretical computer scientists should provide efficient algorithms for problems, not just classify them. Classifications usually help, but they are just a first step. An efficient algorithm is not the same as an algorithm with $O(f(n))$ runtime for a slowly growing $f$:

- Constants may matter.
- Runtime is not the only important parameter.

Implementations very much help to reveal the problems of seemingly efficient algorithms. They lead to better theory.

Automata theory for verification very much profits from "beyond Big-Oh" analysis and prototype implementations.
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Today’s menu

- Appetizer: Universality of finite automata
- Main course: Emptiness of Büchi automata
- Dessert: Universal search

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Universality of finite automata
The problem

Given: a NFA $A$ over alphabet $\Sigma$.
Decide: is $L(A) = \Sigma^*$ ?
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Universality is PSPACE-complete.
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Deterministic algorithm:

Determinize $\rightarrow$ complement $\rightarrow$ check for emptiness.
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Theorem:
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Deterministic algorithm:
Determinize $\rightarrow$ complement $\rightarrow$ check for emptiness.

Complexity:
$O(2^{|A|})$ time and space, and $\Theta(2^{|A|})$ for some family.
End of the story? No!
Subsumption check [DeWDHR06]:

If the powerset construction generates states $Q_1 \subseteq Q_2$, redirect $Q_2$’s incoming arcs to $Q_1$ and remove $Q_2$. 
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- Let $B = \text{Pow}(A)$ (only reachable states).
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- Recall: $L_B(Q) = \bigcup_{q \in Q} L_A(q)$ for every state $Q$ of $B$. 
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- Assume $Q_1 \subseteq Q_2$. We have $L_B(Q_1) \subseteq L_B(Q_2)$ and if $B$ universal then $L_B(Q_1) = L_B(Q_2)$. 
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- Assume \( Q_1 \subseteq Q_2 \). We have \( L_B(Q_1) \subseteq L_B(Q_2) \) and if \( B \) universal then \( L_B(Q_1) = L_B(Q_2) \).
- Let \( B' \) be the result of the operation. Then \( L_{B'} \subseteq L_B \) and if \( B \) universal then \( L_{B'} = L_B \).
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- Let $B'$ be the result of the operation. Then $L_{B'} \subseteq L_B$ and if $B$ universal then $L_{B'} = L_B$.
- So $B'$ universal iff $B$ universal iff $A$ universal.
Potential application to verification

Typical scenario

- System: communicating automata $A_1, A_2, \ldots, A_n$.
- System’s behaviour: automaton $A = A_1 \otimes A_2 \otimes \ldots \otimes A_n$.
- System correct if $L(A) \subseteq L(B)$
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**Usual approach:** $L(A) \subseteq L(B)$ iff $L(A) \cap \overline{L(B)} = \emptyset$
- Compute $A = A_1 \otimes \ldots \otimes A_n$. Possible blowup!
- Check emptiness of $A \times \overline{B}$. 
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**Usual approach:** $L(A) \subseteq L(B)$ iff $L(A) \cap \overline{L(B)} = \emptyset$
- Compute $A = A_1 \otimes \ldots \otimes A_n$. Possible blowup!
- Check emptiness of $A \times \overline{B}$.

**Alternative approach:** $L(A) \subseteq L(B)$ iff $\overline{L(A)} \cup L(B) = \Sigma^*$
- Compute $\overline{A} = \overline{A}_1 \oplus \ldots \oplus \overline{A}_n$.
- Check universality of $A + \overline{B}$. Possible blowup!
Emptiness of Büchi automata
The problem

Given: a Büchi automaton $A$.
Decide: is $L(A) = \emptyset$ ?

Lassos

$A$ is nonempty iff it contains an **accepting lasso**: a path leading from some initial state to some accepting state, followed by a cycle.
## A trivial quadratic algorithm

### The algorithm

1. Compute all reachable final states.
2. For every final state $q$:
   - check if $q$ is reachable from itself.
   - if so, stop and answer “nonempty”.
   - Answer “empty”.
A trivial quadratic algorithm

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Answer “empty”.

Complexity

(1) takes $O(|A|)$ time.
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Complexity

- (1) takes $O(|A|)$ time.
- (2) takes $O(|A|^2)$ time, and there is a family of graphs for which it takes $\Theta(|A|^2)$.
(1) Use DFS to compute a list $\alpha_1, \alpha_2, \ldots, \alpha_k$ of all reachable accepting states.

(2) For $i = 1$ to $k$:
   - use DFS to check if $\alpha_i$ is reachable from itself
   - if so, stop and answer “nonempty”.
   Answer “empty”.
(1) Use DFS to compute a list $\alpha_1, \alpha_2, \ldots, \alpha_k$ of all reachable accepting states sorted in postorder.
(a state is added to list when backtracking from it)

(2) For $i = 1$ to $k$:
- use a modified DFS to check if $\alpha_i$ is reachable from itself without visiting any state reachable from $\alpha_1, \ldots, \alpha_{i-1}$.
- if so, stop and answer “nonempty”.
Answer “empty”.
A first linear algorithm: double-DFS [CVWY91]

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\[ q_0 \rightarrow q_1 \rightarrow q_2 \rightarrow q_3 \rightarrow q_4 \]
\[ q_5 \]

Beyond Big-Oh analysis
Complexity

Time complexity

- Phase (1) takes $O(|A|)$ time.
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In the DFS for $\alpha_i$ we backtrack whenever hitting states visited during the former DFSs, and so every transition is explored at most once.
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- Together: 2 post ops per (reachable) state.
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### Space complexity

For each state we have three possible situations:

- not yet discovered by the first phase;
- discovered by the first, but not yet by the second;
- discovered by both phases.

2 additional bits per (reachable) state.
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If $A$ is nonempty, then the algorithm answers “nonempty”.

Proof:
- Consider the case $k = 2$ (two final states $\alpha_1, \alpha_2$).
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If the algorithm answers “nonempty”, then \( A \) is nonempty. *Easy.*

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**Proof:**
- Consider the case \( k = 2 \) (two final states \( \alpha_1, \alpha_2 \)).
- If some cycle contains \( \alpha_1 \), the algorithm will detect it.

Potential problem: some cycle contains \( \alpha_2 \), some transition of the cycle is reachable from \( \alpha_1 \). Call these cycles blocked.
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- Call these cycles blocked.
Solution: guarantee that if there are blocked cycles, then some cycle contains $\alpha_1$. Because cycles containing $\alpha_1$ are always detected!
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If there is a blocked cycle, then $\alpha_1 \leadsto \alpha_2$.
If $(\alpha_1 \leadsto \alpha_2 \land \alpha_2 \leadsto \alpha_1)$ then some cycle contains $\alpha_1$.
So it suffices to guarantee: if $\alpha_1 \leadsto \alpha_2$ then $\alpha_2 \leadsto \alpha_1$.
We show that postorder implies this.
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If there is a blocked cycle, then $\alpha_1 \rightsquigarrow \alpha_2$.
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We show that postorder implies this.

Look at DFS as a recursive procedure $dfs(q)$.
Let $ca(q)$ denote the time at which $dfs(q)$ is called.
Let $ret(q)$ denote the time at which $dfs(q)$ returns.
(The search backtracks from $q$.)
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Let $ret(q)$ denote the time at which $dfs(q)$ returns.
(The search backtracks from $q$.)
Postorder assumption: $ret(\alpha_1) < ret(\alpha_2)$. 
Lemma

Assume $ret(\alpha_1) < ret(\alpha_2)$. If $\alpha_1 \leadsto \alpha_2$ then $\alpha_2 \leadsto \alpha_1$. 
Lemma

Assume \( \text{ret}(\alpha_1) < \text{ret}(\alpha_2) \). If \( \alpha_1 \rightsquigarrow \alpha_2 \) then \( \alpha_2 \rightsquigarrow \alpha_1 \).

Proof:

By proper nesting of calls we have either:

- \( \text{ca}(\alpha_1) < \text{ret}(\alpha_1) < \text{ca}(\alpha_2) < \text{ret}(\alpha_2) \) or
- \( \text{ca}(\alpha_2) < \text{ca}(\alpha_1) < \text{ret}(\alpha_1) < \text{ret}(\alpha_2) \)
Lemma

Assume \( ret(\alpha_1) < ret(\alpha_2) \). If \( \alpha_1 \leadsto \alpha_2 \) then \( \alpha_2 \leadsto \alpha_1 \).

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- Case 1: \( ca(\alpha_1) < ret(\alpha_1) < ca(\alpha_2) < ret(\alpha_2) \).
  Then \( \alpha_1 \not\leadsto \alpha_2 \).
Lemma

Assume $ret(\alpha_1) < ret(\alpha_2)$. If $\alpha_1 \leadsto \alpha_2$ then $\alpha_2 \leadsto \alpha_1$.

Proof:

By proper nesting of calls we have either:
- $ca(\alpha_1) < ret(\alpha_1) < ca(\alpha_2) < ret(\alpha_2)$ or
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Then $\alpha_1 \not\leadsto \alpha_2$.

Case 2: $ca(\alpha_2) < ca(\alpha_1) < ret(\alpha_1) < ret(\alpha_2)$.
Then $\alpha_2 \leadsto \alpha_1$. 
End of the story? No!
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- Double-DFS requires to explore every transition at least once.
  (Cannot terminate before the end of the first search!)
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- Double-DFS requires to explore every transition at least once.
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- Double-DFS inadequate for producing counterexamples:
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Counterexample: path to accepting state $\alpha_j +$ cycle.
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Double-DFS inadequate for producing counterexamples: Counterexample: path to accepting state $\alpha_j$ + cycle. Double-DFS requires to store paths for all accepting states.
Interleave the two phases.
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At time \( \text{ret}(\alpha_i) \) interrupt the first search and launch the second search for \( \alpha_i \).
Interleave the two phases.
At time $\text{ret}(\alpha_i)$ interrupt the first search and launch the second search for $\alpha_i$.
When the algorithm finds a cycle the call stack contains
- a path to the current final state $\alpha_i$, plus
- a path leading from $\alpha_i$ to itself.
- Interleave the two phases.
- At time $\text{ret}(\alpha_i)$ interrupt the first search and launch the second search for $\alpha_i$.
- When the algorithm finds a cycle the call stack contains
  - a path to the current final state $\alpha_i$, plus
  - a path leading from $\alpha_i$ to itself.
- Counterexample: just pop the call stack!
Interleave the two phases.

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- a path to the current final state \( \alpha_i \), plus
  - a path leading from \( \alpha_i \) to itself.

Counterexample: just pop the call stack!

Correctness: Easy. The second searches are exactly as in the double-DFS algorithm.
End of the story? No!
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Definition
A search algorithm for Büchi emptiness is optimal if it terminates immediately after the set of transitions it has explored contains an accepting lasso.
End of the story? No!

Definition

A search algorithm for Büchi emptiness is \textit{optimal} if it terminates immediately after the set of transitions it has explored contains an accepting lasso.

The nested-DFS algorithm is not optimal!
Minor improvements

[Holzmann, Peled, Yannakakis 96]
If the second search finds a state that is currently in the call stack of the first search, answer “nonempty”.

Beyond Big-Oh analysis
Minor improvements

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If the second search finds a state that is currently in the call stack of the first search, answer “nonempty”.

[Gastin, Moro, Zeitoun 04]
If the first search finds an accepting state that is currently in the call stack, answer “nonempty”.
### Minor improvements

<table>
<thead>
<tr>
<th>Reference</th>
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</tr>
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<tbody>
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<tr>
<td>[Schwoon, E. 05]</td>
<td>These two improvements still require only 2 additional bits per state: four-colour algorithm.</td>
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But: the four-colour algorithm is still not optimal.
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Question
Are there optimal (linear-time) algorithms?
Approach

- Identify the reachable (nontrivial) SCCs of $A$.
- Check if some of them contains an accepting state.
Tarjan’s algorithm for computing SCCs

Basic notions

- Automaton $A \Rightarrow$ dag of SCCs.
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  (The definition of root refers to a particular, fixed DFS-run!)
### Basic notions

- **Automaton** $A \Rightarrow$ dag of SCCs.
- **Root** of a SCC: the first node of the SCC discovered by the DFS.
  (The definition of root refers to a **particular, fixed** DFS-run!)
- If $\rho$ is a root, then at time $\text{ret}(\rho)$ the DFS has discovered all nodes of $\rho$’s SCC and its descendants in the dag.
Basic notions

- Automaton $A \Rightarrow$ dag of SCCs.
- **Root** of a SCC: the first node of the SCC discovered by the DFS.
  (The definition of root refers to a *particular, fixed* DFS-run!)
- If $\rho$ is a root, then at time $\text{ret}(\rho)$ the DFS has discovered all nodes of $\rho$’s SCC and its descendants in the dag.

First idea

- Push all discovered nodes in a new stack (*Tarjan’s stack*).
- For every root $\rho$: at time $\text{ret}(\rho)$, pop from Tarjan’s stack until $\rho$ is popped; the popped nodes constitute $\rho$’s SCC.
GOD’s contribution: Oracle

For a given state $q$ oracle decides if $q$ is a root.

1. $T(q)$
2. push($q$, Stack);
3. for each transition $q \rightarrow r$
4. if $r$ not yet explored then $T(r)$
5. if $q$ is a root then
6. repeat $s := \text{pop}(\text{Stack})$ until $s = q$
Implementing the oracle

Problem
The algorithm must identify the roots of the SCCs. But the SCCs are what we want to compute!
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Second idea
- Annotate each state \( q \) with \( ca(q) \) and a lowlink-number \( lowlink(q) \).
  (Order induced by call numbers is all that matters)
Implementing the oracle

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The algorithm must identify the roots of the SCCs. But the SCCs are what we want to compute!

Second idea

- Annotate each state \( q \) with \( ca(q) \) and a lowlink-number \( \text{lowlink}(q) \).
  
  (Order induced by call numbers is all that matters)

- \( \text{lowlink}(q) \): lowest \( ca(r) \) of states \( r \) satisfying
  - \( q \) and \( r \) lie in the same SCC, and
  - \( r \) reachable from \( q \) through states not yet discovered at time \( ca(q) \).
Implementing the oracle

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The algorithm must identify the roots of the SCCs. But the SCCs are what we want to compute!

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- Annotate each state $q$ with $ca(q)$ and a lowlink-number $lowlink(q)$.
  (Order induced by call numbers is all that matters)
- $lowlink(q)$: lowest $ca(r)$ of states $r$ satisfying
  - $q$ and $r$ lie in the same SCC, and
  - $r$ reachable from $q$ through states not yet discovered at time $ca(q)$.
- $lowlink(q) \leq ca(q)$ for every state $q$. 
Implementing the oracle

Problem

The algorithm must identify the roots of the SCCs. But the SCCs are what we want to compute!

Second idea

- Annotate each state $q$ with $ca(q)$ and a lowlink-number $\text{lowlink}(q)$. 
  (Order induced by call numbers is all that matters)
- $\text{lowlink}(q)$: lowest $ca(r)$ of states $r$ satisfying
  - $q$ and $r$ lie in the same SCC, and
  - $r$ reachable from $q$ through states not yet discovered at time $ca(q)$.
- $\text{lowlink}(q) \leq ca(q)$ for every state $q$.
- Fact: $\text{lowlink}(q) = ca(q)$ if and only if $q$ is a root.
Miracle

\[ \text{lowlink}(q) \text{ can be easily determined at time } ret(q). \]
Tarjan’s algorithm

**Miracle**

\( \text{lowlink}(q) \) can be easily determined at time \( \text{ret}(q) \).

1. T(q)
2. push(q, Stack);
3. for each transition \( q \rightarrow r \)
4. if \( r \) not yet explored then
5. T(r);
6. \( r.\text{lowlink} := \min(q.\text{lowlink}, r.\text{lowlink}) \)
7. else if \( r \in \text{Stack} \) then
8. \( r.\text{lowlink} := \min(q.\text{lowlink}, r.ca) \)
9. if \( q.\text{lowlink} = q.ca \) then
10. repeat \( s := \text{pop}(\text{Stack}) \) until \( s = q \)
A direct modification of Tarjan's algorithm for emptiness checking is non-optimal.
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Requires to completely explore an SCC before it is popped from the stack.

Main observation of [GV04]:
\( \alpha \) belongs to a cycle iff \( T(\alpha) \) reaches some state \( r \) satisfying two conditions:
- \( r \in \text{Stack} \), and
- \( \text{lowlink}(r) < \text{ca}(\alpha) \).
Add a new parameter to the procedure to keep track of the last visited accepting state.

\begin{algorithm}
  \textbf{GV}(q, \alpha)
  \begin{algorithmic}
    \State push\((q, Stack)\);
    \For {each transition \( q \rightarrow r \)}
      \If {\( r \) not yet explored}
        \If {\( r \) accepting}
          \State \textbf{GV}(r, r)
        \Else
          \State \textbf{GV}(r, \alpha);
        \EndIf
        \State \( r\.lowlink := \min(q\.lowlink, r\.lowlink) \)
      \EndIf
      \ElseIf {\( r \in Stack \)}
        \If {\( r\.lowlink < \alpha.ca \)}
          \State report “nonempty”;
        \EndIf
        \State \( r\.lowlink := \min(q\.lowlink, r.ca) \)
      \EndElseIf
    \EndFor
    \If {\( q\.lowlink = q.ca \)}
      \Repeat
        \State \( s := \text{pop}(Stack) \)
      \Until {\( s = q \)}
  \end{algorithmic}
\end{algorithm}
End of the story? **No!**
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Time complexity

[GV04] requires only one post op per state.
End of the story? **No!**

### Time complexity

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### Space complexity

- [GV04] requires to store two numbers per state plus a third number for Tarjan’s stack ($3 \cdot \log n$ bits per state).
End of the story? No!

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- [GV04] requires to store two numbers per state plus a third number for Tarjan’s stack (3 \cdot \log n \text{ bits per state}).
- Compare with 2 bits per state of nested-DFS or the four-colour algorithm.
End of the story? No!

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**Generalized Büchi automata**

- LTL \( \rightarrow \) Büchi translations yield generalized BA.
End of the story? No!

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Generalized Büchi automata

- LTL $\rightarrow$ Büchi translations yield generalized BA.
- GBA with $n$ states and $k$ acceptings sets $\rightarrow$ BA with $n \cdot k$ states. Expensive!
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Generalized Büchi automata

- LTL → Büchi translations yield generalized BA.
- GBA with \(n\) states and \(k\) acceptings sets → BA with \(n \cdot k\) states. Expensive!
- Neither nested-DFS nor GV can be extended to GBA.
Do optimal algorithms exist that
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- require less memory, and
Do optimal algorithms exist that
- require less memory, and
- can be easily extended to GBAs?
First idea

Partition Stack into **Roots** and **Nonroots**, keeping the following invariant:
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Partition Stack into **Roots** and **Nonroots**, keeping the following invariant:

- **Roots** contains all nodes that are roots of the part of the graph explored so far.
- **Nonroots**: contains all nodes that are non-roots of the part of the graph explored so far.
Couvreur and Gabow’s algorithm [C99,G00]

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Key insight: \( q \) is a root iff it is a root of the part of the graph explored at time \( \text{ret}(q) \).
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Key insight: $q$ is a root iff it is a root of the part of the graph explored at time $ret(q)$.

So we can check if $q$ is a root by checking $q = \text{top(}\text{Roots)}$ at time $ret(q)$.
First idea

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- **Roots**: contains all nodes that are roots of the part of the graph explored so far.
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Key insight: *q* is a root iff it is a root of the part of the graph explored at time \( \text{ret}(q) \).

So we can check if *q* is a root by checking *q* = \( \text{top}(\text{Roots}) \) at time \( \text{ret}(q) \).

New problem: to keep the invariant.
GOD’s contribution: oracle to keep the invariant

For $q \rightarrow r$, the oracle decides if $q$ reachable from $r$: $r \rightsquigarrow q$. 
GOD’s contribution: oracle to keep the invariant

- For $q \rightarrow r$, the oracle decides if $q$ reachable from $r$: $r \leadsto q$. 

Observe: if $r \leadsto q$ then $r$ belongs to a cycle.

We show: no node in Roots discovered after $r$ can be a root.
GOD’s contribution: oracle to keep the invariant

- For $q \rightarrow r$, the oracle decides if $q$ reachable from $r$: $r \simto q$.
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- For $q \rightarrow r$, the oracle decides if $q$ reachable from $r$: $r \rightsquigarrow q$.
- Observe: if $r \rightsquigarrow q$ then $r$ belongs to a cycle.
- We show: no node in Roots discovered after $r$ can be a root.
GCG(q)
  push(q, Roots);
  for each transition q → r
    if r not yet explored then GCG(r)
  elseif r ⇝ q then
    repeat
      s := pop(Roots); push(Nonroots);
      if s is accepting report “nonempty”
      until ca(s) ≤ ca(r);
    push(s, Roots); pop(Nonroots)
  if top(Roots) = q then
    pop(Roots);
  while ca(top(Nonroots)) > ca(q)
    pop(Nonroots)
Example

Javier Esparza
Beyond Big-Oh analysis
Correctness I

If $s$ is popped \textbf{at line 7}, then it belongs to a cycle containing $r$. 
Correctness and optimality

Correctness I

If $s$ is popped at line 7, then it belongs to a cycle containing $r$.

Proof:

- Situation: $q \rightarrow r \rightsquigarrow q$, $s \in \text{Roots}$, $ca(s) > ca(r)$.
Correctness and optimality

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If $s$ is popped at line 7, then it belongs to a cycle containing $r$.

Proof:
- Situation: $q \rightarrow r \sim q$, $s \in \text{Roots}$, $ca(s) > ca(r)$.
- We show $\rho_r \sim s \sim q \rightarrow r \sim \rho_r$. 
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- We show $\rho_r \rightsquigarrow s \rightsquigarrow q \rightarrow r \rightsquigarrow \rho_r$.
- $s$ is a DFS-ascendant of $q$, and so $s \rightsquigarrow q$. 
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- $s$ is a DFS-ascendant of $q$, and so $s \rightsquigarrow q$.
  Because $s \in Roots$, and $Roots$ subset of DFS-stack.
**Correctness and optimality**

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- We show $\rho_r \sim s \sim q \rightarrow r \sim \rho_r$.
- $s$ is a DFS-ascendant of $q$, and so $s \sim q$.
  - Because $s \in Roots$, and $Roots$ subset of DFS-stack.
- $\rho_r$ is a DFS-ascendant of $s$, and so $\rho_r \sim s$. 

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- $s$ is a DFS-ascendant of $q$, and so $s \rightsquigarrow q$.
- Because $s \in Roots$, and Roots subset of DFS-stack.
- $\rho_r$ is a DFS-ascendant of $s$, and so $\rho_r \rightsquigarrow s$.
- Since $q \rightarrow r \rightsquigarrow q$, we have $\rho_r = \rho_q$, and so $\rho_r$ is a DFS-ascendant of $q$. 

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Correctness I
If \( s \) is popped at line 7, then it belongs to a cycle containing \( r \).

Proof:
- Situation: \( q \rightarrow r \leadsto q, s \in \text{Roots}, ca(s) > ca(r) \).
- We show \( \rho_r \leadsto s \leadsto q \rightarrow r \leadsto \rho_r \).
- \( s \) is a DFS-ascendant of \( q \), and so \( s \leadsto q \).
  Because \( s \in \text{Roots} \), and \( \text{Roots} \) subset of DFS-stack.
- \( \rho_r \) is a DFS-ascendant of \( s \), and so \( \rho_r \leadsto s \).
  Since \( q \rightarrow r \leadsto q \), we have \( \rho_r = \rho_q \), and so \( \rho_r \) is a DFS-ascendant of \( q \).
  So either \( \rho_r \) is DFS-ascendant of \( s \) or vice versa.
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  Because \( s \in \text{Roots} \), and \( \text{Roots} \) subset of DFS-stack.
- \( \rho_r \) is a DFS-ascendant of \( s \), and so \( \rho_r \leadsto s \).
  Since \( q \rightarrow r \leadsto q \), we have \( \rho_r = \rho_q \), and so \( \rho_r \) is a DFS-ascendant of \( q \).
  So either \( \rho_r \) is DFS-ascendant of \( s \) or vice versa.
  But \( s \) cannot be a DFS-ascendant of \( \rho_r \) because 
  \( ca(\rho_r) \leq ca(r) < ca(s) \).
Correctness and optimality

Correctness II

If a state $s$ is popped at line 7 and $ca(s) > ca(r)$, then it is not a root.
Correctness and optimality

Correctness II

If a state $s$ is popped at line 7 and $ca(s) > ca(r)$, then it is not a root.

Proof:
- $s$ belongs to a cycle containing $r$, and, since $ca(s) > ca(r)$, it is not a root.
Correctness and optimality

<table>
<thead>
<tr>
<th>Correctness III + Optimality</th>
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<tbody>
<tr>
<td>Every reachable state $q$ belonging to some cycle is eventually popped at line 7.</td>
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<td>Moreover, $q$ is popped immediately after any cycle containing it is completely explored.</td>
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Correctness and optimality

Correctness III + Optimality

Every reachable state $q$ belonging to some cycle is eventually popped at line 7.
Moreover, $q$ is popped immediately after any cycle containing it is completely explored.

Proof:

- Fix a cycle $C$ containing $q$. 
Correctness and optimality

Correctness III + Optimality

Every reachable state $q$ belonging to some cycle is eventually popped at line 7. Moreover, $q$ is popped immediately after any cycle containing it is completely explored.

Proof:

1. Fix a cycle $C$ containing $q$.
2. Let $r$ be the last successor of $q$ along $C$ such that at time $ca(q)$ there is a path of unexplored states from $q$ to $r$ (count $q$ as unexplored, possibly $q = r$).
Correctness and optimality

Correctness III + Optimality

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- Fix a cycle \( C \) containing \( q \).
- Let \( r \) be the last successor of \( q \) along \( C \) such that at time \( ca(q) \) there is a path of unexplored states from \( q \) to \( r \) (count \( q \) as unexplored, possibly \( q = r \)).
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Correctness and optimality

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- Let $r$ be the last successor of $q$ along $C$ such that at time $ca(q)$ there is a path of unexplored states from $q$ to $r$ (count $q$ as unexplored, possibly $q = r$).
- Let $s$ be the successor of $r$ along $C$.
- $ca(s) \leq ca(q) \leq ca(r)$, and so $ca(s) \leq ca(r)$. 

Javier Esparza  Beyond Big-Oh analysis
Correctness and optimality

Correctness III + Optimality

Every reachable state $q$ belonging to some cycle is eventually popped at line 7. Moreover, $q$ is popped immediately after any cycle containing it is completely explored.

Proof:

- Fix a cycle $C$ containing $q$.
- Let $r$ be the last successor of $q$ along $C$ such that at time $ca(q)$ there is a path of unexplored states from $q$ to $r$ (count $q$ as unexplored, possibly $q = r$).
- Let $s$ be the successor of $r$ along $C$.
- $ca(s) \leq ca(q) \leq ca(r)$, and so $ca(s) \leq ca(r)$.
- So $q$ is popped at line 7 when $q \rightarrow r$ is explored, or earlier.
## Correctness and optimality

### Correctness III

Every state discovered by the search and not belonging to any cycle is eventually popped at line 12.

### Proof:

Easy.
Implementing the oracle

Lemma

Asume the oracle is asked at time $t$ whether $r \rightsquigarrow q$. The answer is “yes” iff $t < ret(\rho_r)$. 
Implementing the oracle

Lemma
Asume the oracle is asked at time $t$ whether $r \rightsquigarrow q$.
The answer is “yes” iff $t < ret(\rho r)$.

Proof:
- Situation: $ca(q) \leq t < ret(q)$, $q \rightarrow r$, $ca(r) \leq t$. 
Implementing the oracle

**Lemma**

Assume the oracle is asked at time $t$ whether $r \leadsto q$. The answer is “yes” iff $t < \text{ret}(\rho_r)$.

**Proof:**

- Situation: $\text{ca}(q) \leq t < \text{ret}(q)$, $q \rightarrow r$, $\text{ca}(r) \leq t$.
- Assume $r \leadsto q$. If $t \geq \text{ret}(\rho_r)$, then $t \geq \text{ret}(q)$, contradiction. So $t < \text{ret}(\rho_r)$.
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Lemma

Assume the oracle is asked at time $t$ whether $r \leadsto q$. The answer is “yes” iff $t < \text{ret}(\rho_r)$.

Proof:

- Situation: $ca(q) \leq t < \text{ret}(q)$, $q \rightarrow r$, $ca(r) \leq t$.
- Assume $r \leadsto q$. If $t \geq \text{ret}(\rho_r)$, then $t \geq \text{ret}(q)$, contradiction. So $t < \text{ret}(\rho_r)$
- Assume $r \not\leadsto q$. Then $q \leadsto \rho_r \not\leadsto q$. 
Lemma

Asume the oracle is asked at time $t$ whether $r \rightsquigarrow q$. The answer is “yes” iff $t < \text{ret}(\rho_r)$.

Proof:

- Situation: $\text{ca}(q) \leq t < \text{ret}(q)$, $q \rightarrow r$, $\text{ca}(r) \leq t$.
- Assume $r \rightsquigarrow q$. If $t \geq \text{ret}(\rho_r)$, then $t \geq \text{ret}(q)$, contradiction. So $t < \text{ret}(\rho_r)$
- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.
  By postorder lemma, $\text{ret}(\rho_r) < \text{ret}(q)$. 
Implementing the oracle

Lemma

Asume the oracle is asked at time $t$ whether $r \leadsto q$.
The answer is “yes” iff $t < ret(\rho_r)$.

Proof:

- Situation: $ca(q) \leq t < ret(q)$, $q \rightarrow r$, $ca(r) \leq t$.
- Assume $r \leadsto q$. If $t \geq ret(\rho_r)$, then $t \geq ret(q)$, contradiction. So $t < ret(\rho_r)$
- Assume $r \not\leadsto q$. Then $q \leadsto \rho_r \not\leadsto q$.
  By postorder lemma, $ret(\rho_r) < ret(q)$.
  Case 1: $ca(\rho_r) < ret(\rho_r) < ca(q) \leq t < ret(q)$. Done.
Implementing the oracle

Lemma

Assume the oracle is asked at time $t$ whether $r \rightsquigarrow q$. The answer is “yes” iff $t < \text{ret}(\rho_r)$.

Proof:

- Situation: $ca(q) \leq t < \text{ret}(q)$, $q \rightarrow r$, $ca(r) \leq t$.
- Assume $r \rightsquigarrow q$. If $t \geq \text{ret}(\rho_r)$, then $t \geq \text{ret}(q)$, contradiction. So $t < \text{ret}(\rho_r)$.
- Assume $r \not\rightsquigarrow q$. Then $q \rightsquigarrow \rho_r \not\rightsquigarrow q$.
  By postorder lemma, $\text{ret}(\rho_r) < \text{ret}(q)$.
  Case 1: $ca(\rho_r) < \text{ret}(\rho_r) < ca(q) \leq t < \text{ret}(q)$. Done.
  Case 2: $ca(q) < ca(\rho_r) \leq ca(r) < \text{ret}(\rho_r) < \text{ret}(q)$.
  Since at time $t$ we are executing $dfs(q)$, we have $\text{ret}(\rho_r) < t \leq \text{ret}(q)$. 

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Implementing the oracle

**Lemma**

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Idea

- Recall $ca(r) \leq t$.
- At time $ret(\rho)$ removes all nodes from $\rho$’s SCC from Rots and Nonroots.
- So $r$ stays in Stack exactly during the interval $[ca(r), ret(root(t))]$, and therefore:
Implementing the oracle

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Asume the oracle is asked at time $t$ whether $r \rightsquigarrow q$. The answer is “yes” iff $t < \text{ret}(\rho_r)$.

Idea
- Recall $ca(r) \leq t$.
- At time $\text{ret}(\rho)$ removes all nodes from $\rho$’s SCC from Rots and Nonroots.
- So $r$ stays in Stack exactly during the interval $[ca(r), \text{ret}(\text{root}(t))]$, and therefore:
  
  $$t < \text{ret}(\rho_r) \text{ iff } r \in \text{Roots} \cup \text{Nonroots} \text{ at time } t.$$
1 GCG(q)
2     push(q, Roots);
3     for each transition q → r
4         if r not yet explored then GCG(r)
5         elseif r ∈ Roots ∪ Nonroots then
6             repeat
7                 s := pop(Roots); push(Nonroots);
8                 if s is accepting report “nonempty”
9                 until ca(s) ≤ ca(r);
10             push(s, Roots); pop(Nonroots)
11     if top(Roots) = q then
12         pop(Roots);
13     while ca(top(Nonroots)) > ca(q)
14         pop(Nonroots)
Extension to generalized Büchi automata

Store for each state \( q \in Roots \) a subset \( q.acc \) of accepting sets, maintaining the following invariant:

- \( q.acc \) contains all the accepting sets intersecting \( q \)'s SCC in the part of the graph explored so far.
Extension to generalized Büchi automata

Store for each state $q \in \text{Roots}$ a subset $q.\text{acc}$ of accepting sets, maintaining the following invariant:

- $q.\text{acc}$ contains all the accepting sets intersecting $q$’s SCC in the part of the graph explored so far.

When GC($q$) pops a cycle, add all the acc’s of the popped states to $q.\text{acc}$.
1. EGC(q)
2. push(q, Roots);
3. \textbf{q.}acc := accepting sets containing q;
4. for each transition q \rightarrow r
5. \textbf{if} r \textbf{not yet explored then} EGC(r)
6. \textbf{elseif} r \in \textbf{Roots} \cup \textbf{Nonroots} \textbf{then}
7. \textbf{repeat}
8. \textbf{s :=} pop(\textbf{Roots}); push(s, \textbf{Nonroots});
9. \textbf{q.}acc := q.\textbf{acc} \cup s.\textbf{acc}
10. \textbf{until} \textbf{ca}(s) \leq \textbf{ca}(r);
11. push(s, \textbf{Roots}); \textbf{pop(Nonroots)};
12. \textbf{if} q.\textbf{acc} = \textbf{all accepting sets report “nonempty”}
13. \textbf{if} q = \textbf{top(Roots)} \textbf{then}
14. \textbf{pop(Roots)};
15. \textbf{while} \textbf{ca(top(Nonroots)}) > \textbf{ca}(q)
16. \textbf{pop(Nonroots)}
The SCC of a root can also be determined as follows:

- Introduce one extra bit $b_q$ for every state $q$. Initially $b_q = 0$.
- For every root $\rho$: at time $ret(\rho)$ conduct a DFS to set to 1 the bits of all states reachable from $\rho$.
- The set of states that had to be flipped constitute $\rho$’s SCC.
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- The set of states that had to be flipped constitute $\rho$’s SCC.

Gets rid of Nonroots, but requires one extra DFS.
End of the story? No!

Černá and Pelánek’s observation [ČP03]

- Many LTL specifications are translated into weak Büchi automata.
- The four-colour algorithm without the second search is correct for weak automata.
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Schwoon and E. [SE05]
The four-colour algorithm without the second searches is optimal for weak automata.
## End of the story?

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### Practical relevance of differences in space complexity
- Small when state descriptors explicitly stored. (state descriptors are often dozens of bytes long)
- Large when state-hashing is applied. (one or two bits for storing a state)
Open questions

- Are there optimal algorithms requiring only a constant number of additional bits per state?
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- Are there algorithms for GBA requiring only a constant number of additional bits per state?
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- Are there algorithms for GBA requiring only a constant number of additional bits per state?
- Can a shortest counterexample be computed in linear time?
Universal search
Introduced by Levin.
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Used here as a theoretical justification of the need for going beyond Big-Oh analysis.
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Intuitively . . .
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Intuitively . . .

Let $A[x]$ be an algorithm computing $F(x)$ in $f(n)$ time. $A$ is optimal for $F$ if no other algorithm computes $F$ in $o(f(n))$ time.
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We give a **universal algorithm** that is optimal **for every** $F$. 
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We give a universal algorithm that is optimal for every $F$.

Corollary: if constants don’t matter we are all useless!
A bit more formally . . .

- Fix a formal system (i.e., ZF).
- A function is provably computable if some algorithm computes it and the algorithm’s correctness is a theorem of the system.
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- Fix a formal system (i.e., ZF).
- A function is **provably computable** if some algorithm computes it and the algorithm’s correctness is a theorem of the system.

**Theorem (Levin)**

There is an algorithm $U[F, x]$ such that $U[F, –]$ is optimal for every provably computable function $F$. 
A non-optimal algorithm $U_1[F,-]$

We describe first an obviously correct algorithm $U_1[F,-]$. On input $x$, $U_1[F,-]$ behaves as follows:

- $U_1[F,-]$ enumerates all pairs $\Pi = (P, D)$, where $P$ program and $D$ derivation of the formal system. Let $\Pi_1, \Pi_2, \Pi_3 \ldots$ be this enumeration.

- For every $\Pi_i = (P_i, D_i)$: $U_1[F,-]$ checks if $D_i$ is a proof that $P_i$ computes $F$. If so, $U_1[F,-]$ computes $P_i[x]$ and stops.
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The algorithm $U[F, –]$

$U[F, x]$ dovetails the computations of $U_1[F, –]$. It spends:
- every second step on $\Pi_1$;
- every second step of the remaining ones on $\Pi_2$;
- every second step of the remaining ones on $\Pi_3$, etc.
Claim

If $P$ runs in $f(n)$ time, then $U[F, -]$ runs in $O(f(n))$ time.
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Proof idea:

Let $i$ be the smallest index such that $P_i = P$ and $D_i$ proves that $P$ computes $F$. (Observe: $i$ independent of $x$!)
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Proof idea:
Let $i$ be the smallest index such that $P_i = P$ and $D_i$ proves that $P$ computes $F$. (Observe: $i$ independent of $x$!) Then $U[F, -]$ terminates on input $x$ after executing $f(x)$ steps of $\Pi_i$, or earlier.
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Total number of steps executed by \( U[F, -] \) on \( x \):

So \( U[F, -] \) takes at most \( 2^{i+1} \cdot f(x) = O(f(x)) \) steps.
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- Steps spent on $\Pi_i, \Pi_{i-1}, \ldots, \Pi_1$:
  \[ f(x) + 2f(x) + 2^2f(x) + \ldots + 2^i f(x) = (2^{i+1} - 1)f(x) \]

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  \]

- Steps spent on $\Pi_{i+1}, \Pi_{i+2}, \ldots$:
  
  \[
  \frac{1}{2}f(x) + \frac{1}{4}f(x) + \ldots + 1 \leq f(x) = f(x)
  \]

So $U[F, –]$ takes at most $2^{i+1} \cdot f(x) = O(f(x))$ steps.
Conclusions

Going beyond Big-Oh analysis in verification is important. It is not only about heuristics and hacking: good theory is waiting for us there.

Javier Esparza
Beyond Big-Oh analysis
Conclusions

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