Logics for Continuous Reachability in Petri Nets and Vector Addition Systems with States

Michael Blondin

Christoph Haase
Petri nets

Places

Transitions

Arcs

Tokens

Disabled

Enabled

Reachability:

\[ u \rightarrow N \rightarrow v \]

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)
Petri nets

Places

Reachability: EXPSPACE-hard (Lipton '76)
Cubic-Ackermannian (Leroux and Schmitz LICS'15)
Petri nets

- Places
- Transitions
- Arcs
- Tokens
- Disabled
- Enabled

Reachability: $u \rightarrow N \rightarrow v$

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

Transitions
Petri nets

Places
Transitions
Arcs
Tokens
Disabled
Enabled

(2, 0, 0)

Reachability: $u \Rightarrow v$

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

1/10
Petri nets

Places

Transitions

Arcs

Tokens

Reachability: $u \Rightarrow N \Rightarrow v$

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

Tokens
Petri nets

Places

Transitions

Arcs

Tokens

Disabled

Reachability:

$N(2, 1, 0)$

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

Disabled
Petri nets

Reachability: EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

(2, 0, 0)

Enabled
Petri nets

Reachability: EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

Places
Transitions
Arcs
Tokens
Disabled
Enabled

(2, 0, 0)

\[ N \]

\[ N \]

\[ N \]
Petri nets

Reachability: EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

Places

Transitions

Arcs

Tokens

Disabled

Enabled
Petri nets

Places

Transitions

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Tokens

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Enabled

(2, 0, 0)

Reachability:

\[ u \rightarrow N \rightarrow v \]

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)
Petri nets

Places
Transitions
Arcs
Tokens
Disabled
Enabled

\[(2, 0, 0) \overset{*}{\longrightarrow}_\mathbb{N} (2, 1, 0)\]
Reachability: $u \xrightarrow{\ast}^N v$?
Petri nets

Concurrent programs
Protocols
Business processes
Biological processes

Reachability: $u \xrightarrow{*}^N v$?

correct?
Verifying multi-threaded programs

Counters:  number of threads
States:  shared resources
Petri nets / Vector addition systems with states (VASS)

Reachability:

Reachability:

Reachability:

Reachability:

Reachability:
Petri nets / Vector addition systems with states (VASS)

Places: 
- (0, 1, -1)
- (0, 0, 0)
- (0, -1, 2)

Transitions: 
- p(0, 0, 1) \rightarrow N p(1, 0, 2)
- p(x, y, z) \rightarrow 0 < y + z \leq 2x

Control states:

Reachability: 
- p(u) \rightarrow N q(v)
- \frac{1}{10}

EXPSPACE-hard (Lipton '76)
Cubic-Ackermannian (Leroux and Schmitz LICS'15)
Petri nets / Vector addition systems with states (VASS)

Transitions

(p) (q)

(0, 1, -1) (0, 0, 0) (0, -1, 2)

(0, 1, -1) (0, 0, 0) (0, -1, 2)

Reachability: \( u \xrightarrow{} N \ x \xrightarrow{} v \)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)

EXPSPACE-hard (Lipton '76)

Reachability: \( p \xrightarrow{} q \)
Petri nets / Vector addition systems with states (VASS)

Places: $p$, $q$

Transitions: $(0, 1, -1)$, $(0, 0, 0)$, $(0, -1, 2)$

Arcs: $(0, 1, -1) \rightarrow (0, 0, 0)$, $(0, 0, 0) \rightarrow (0, -1, 2)$, $(0, -1, 2) \rightarrow (0, 0, 0)$

Tokens:
- $(0, 0, 1)$
- $(0, 0, 2)$

Reachability:
- $p(0, 0, 1)$
- $q(0, 0, 2)$

Control-states: $p(0, 0, 1)$

Reachability:
- $p(u) \rightarrow q(v)$ for $0 < y + z < 2x$
Petri nets / Vector addition systems with states (VASS)

\[(0, 1, -1) \rightarrow (0, 0, 0) \rightarrow (0, -1, 2) \rightarrow (1, 0, 0) \rightarrow (0, 0, 1)\]
Petri nets / Vector addition systems with states (VASS)

Reachability:

\[ p(0, 1, 0) \]

\[ (0, 1, -1) \] \[ (0, 0, 0) \] \[ (0, -1, 2) \]

\[ (1, 0, 0) \]
Petri nets / Vector addition systems with states (VASS)

Reachability: $p(0, 1, 0)$

Places

Transitions

Arcs

Tokens

Disabled

Enabled

$(0, 1, -1)$

$(0, 0, 0)$

$(0, -1, 2)$

$(1, 0, 0)$

Reachability: $p(u) \not\rightarrow N q(v)$

EXPSPACE-hard (Lipton '76)

Cubic-Ackermannian (Leroux and Schmitz LICS'15)
Petri nets / Vector addition systems with states (VASS)

Places, Transitions, Arcs, Tokens

Reachability:

$(0, 1, -1)$  $(0, 0, 0)$  $(0, -1, 2)$

$p(0, 1, 0)$

Control-states

Transitions

$p(0, 0, 1)$  $p(x, y, z)$

Reachability:

$p(0, 1, 0)$

$0 < y + z < 2x$
Petri nets / Vector addition systems with states (VASS)

Reachability: \( p(0, 0, 1) \) \( \not\rightarrow N \) \( p(0, 0, 1) \)

Reachability: \( p(u) \not\rightarrow N q(v) \)

\( \frac{1}{10} \)

- \((0, 1, -1)\)
- \((0, 0, 0)\)
- \((0, -1, 2)\)
- \((1, 0, 0)\)

\( q(0, 1, 0) \)
Petri nets / Vector addition systems with states (VASS)

Places: $p$, $q$

Transitions: $p(0, 0, 1) \rightarrow N p(1, 0, 2)$, $q(0, 1, 0)$

Arcs: $p \rightarrow (0, 1, -1)$, $(0, 0, 0) \rightarrow q(0, -1, 2)$

Tokens: $p(0, 0, 1)$

Reachability: $p(u) \rightarrow N q(v)$

Expression: $0 < y + z \leq 2x$

Lipton '76: EXPSPACE-hard

Leroux and Schmitz LICS'15: Cubic-Ackermannian
Petri nets / Vector addition systems with states (VASS)

Reachability: $p(u) \rightarrow q(v) \frac{1}{10}$
Petri nets / Vector addition systems with states (VASS)

Places: $p$, $q$

Transitions: $p(0, 0, 1) \rightarrow p(1, 0, 2)$, $q(0, 0, 2) \rightarrow q(0, 0, 2)$

Arcs: $p(0, 1, -1) \rightarrow q(0, 0, 0)$, $q(0, 0, 0) \rightarrow p(1, 0, 0)$, $p(0, 0, 0) \rightarrow q(0, -1, 2)$

Reachability:

$q(0, 0, 2)$

$(0, 1, -1)$, $(0, 0, 0)$, $(0, -1, 2)$
Petri nets / Vector addition systems with states (VASS)

Reachability:

$\text{Reachability: } p(u) \Rightarrow N q(v)$

$\text{EXPSPACE-hard (Lipton '76)}$

$\text{Cubic-Ackermannian (Leroux and Schmitz LICS'15)}$

Control-states:

$p(0, 0, 1) \Rightarrow N p(1, 0, 2)$

$p(0, 0, 1) \Rightarrow N p(x, y, z)$

$0 < y + z < 2x$

$p(1, 0, 2)$
Petri nets / Vector addition systems with states (VASS)

Reachability:

\[ p(0, 0, 1) \xrightarrow{N} p(1, 0, 2) \]
Petri nets / Vector addition systems with states (VASS)

\[(0, 1, -1) \xrightarrow{p} (0, 0, 0) \xrightarrow{p} (0, -1, 2)\]

\[\text{Reachability: } p(u) \rightarrow p(v) \quad \text{for } 0 < y + z \leq 2^x\]
Reachability: $p(u) \xrightarrow{\mathbb{N}} q(v)$?
Reachability is...

- equivalent for Petri nets and VASS
- not expressible in $\text{FO}(\mathbb{N}, +, <)$
- EXPSPACE-hard (Lipton '76)
- solvable in cubic-Ackermannian time (Leroux and Schmitz LICS'15)
Continuous Petri nets

Can fire transitions fractionally
Continuous Petri nets

Can fire transitions fractionally

David & Alla '87
Continuous Petri nets

Can fire transitions fractionally

...but NP-complete

...but no control-states

Fast with SMT solver
Continuous Petri nets

Can fire transitions fractionally

\( (2, 0, 0) \rightarrow Q + (1\frac{1}{2}, 1, \frac{1}{2}) \)

Fast with SMT solver...but NP-complete...but no control-states

2/10
Can fire transitions fractionally
Continuous Petri nets

Can fire transitions fractionally.

\((2, 0, 0) \rightarrow Q_+ (1\frac{1}{2}, 1, \frac{1}{2})\)
Continuous Petri nets

Can fire transitions fractionally: \((2, 0, 0)\)

\[ Q + (1.5, 1, 0.5) \]

Continuous reachability: \( u \rightarrow_q Q^+ v \)?
Continuous reachability is...

- an over-approximation:
  \[ \neg (u \xrightarrow{*} \mathbb{Q}_+ v) \text{ implies } \neg (u \xrightarrow{*} \mathbb{N} v) \]

- PTIME-complete  
  (Fraca and Haddad PN'13)

- expressible in \( \exists \text{FO}(\mathbb{Q}_+, +, <) \)  
  (B., Finkel, Haase and Haddad TACAS'16)
Continuous reachability is...

- an over-approximation

  fast with SMT solver

  ...but NP-complete

- PTIME-complete

- expressible in $\exists \mathbf{FO} (\mathbb{Q}_+, +, <)$
Continuous reachability is...

- an over-approximation
  - often good
  - ...but no control-states
- PTIME-complete
- expressible in $\exists \text{FO}(\mathbb{Q}_+, +, <)$
new fragment of $\exists \mathsf{FO}(\mathbb{Q}_+, +, <)$

- PTIME-complete
- equivalent to Petri net continuous reachability

new model: continuous VASS

- with control-states
- with reachability equivalent to $\exists \mathsf{FO}(\mathbb{Q}, +, <)$
Our contribution

new fragment of $\exists \text{FO}(\mathbb{Q}_+, +, <)$

- PTIME-complete
- equivalent to Petri net continuous reachability

new model: continuous VASS

- with control-states
- with reachability equivalent to $\exists \text{FO}(\mathbb{Q}, +, <)$
Expressing continuous reachability

\[ y \text{ reachable} \iff \exists \text{ Parikh vector } \pi \text{ s.t.} \]
Expressing continuous reachability

\[ \begin{align*}
\mathbf{y}(p) &= 2 - \pi(s) + \pi(t) \\
\mathbf{y}(q) &= \pi(s) \\
\mathbf{y}(r) &= \pi(s) + \pi(t)
\end{align*} \]

\( \mathbf{y} \) reachable \( \iff \exists \) Parikh vector \( \pi \) s.t.
Expressing continuous reachability

\[\begin{align*}
   p & \rightarrow s \\
   s & \rightarrow q \\
   s & \rightarrow t \\
   t & \rightarrow r \\
   r & \rightarrow q
\end{align*}\]

**y** reachable \iff \exists Parikh vector \( \pi \) s.t.

b) \( \{ u : \pi(u) > 0 \} \) firable from \((2, 0, 0)\)

c) \( \{ u : \pi(u) > 0 \} \) firable to **y**
Expressing continuous reachability

linear size $\exists FO(\mathbb{Q}_+, +, <)$ formula

b) \{u : \pi(u) > 0\} firable from (2, 0, 0)

c) \{u : \pi(u) > 0\} firable to $y$
Expressing continuous reachability

linear size $\exists FO(Q_+, +, <)$ formula

b) $\{u : \pi(u) > 0\}$ firable from $(2, 0, 0)$

c) $\{u : \pi(u) > 0\}$ firable to $y$
Expressing continuous reachability

linear size $\exists FO(Q_+, +, <)$ formula

b) $\{u : \pi(u) > 0\}$ firable from $(2, 0, 0)$
c) $\{u : \pi(u) > 0\}$ firable to $y$
Expressing continuous reachability

linear size $\exists FO(Q_+, +, \prec)$ formula

b) $\{ u : \pi(u) > 0 \}$ firable from $(2, 0, 0)$

c) $\{ u : \pi(u) > 0 \}$ firable to $y$
Expressing continuous reachability

linear size $\exists F\forall (Q_+, +, <)$ formula

b) $\{u : \pi(u) > 0\}$ firable from (2, 0, 0)

c) $\{u : \pi(u) > 0\}$ firable to $y$
Expressing continuous reachability

\[ \exists \text{FO}(\mathbb{Q}_+, +, \prec) \text{ formula} \]

NP-complete
Conjunction of terms of the form

\[ a_1 \cdot x_1 + \ldots + a_n \cdot x_n \sim c \lor \bigvee_{1 \leq i \leq m} \bigwedge_{j \in J_i} x_j > 0 \]
Conjunction of terms of the form

\[ a_1 \cdot x_1 + \ldots + a_n \cdot x_n \sim c \lor \bigvee_{1 \leq i \leq m} \bigwedge_{j \in J_i} x_j > 0 \]

variables over \( \mathbb{Q}_+ \)
Convex linear Horn constraints

Conjunction of terms of the form

\[ a_1 \cdot x_1 + \ldots + a_n \cdot x_n \sim c \lor \bigvee_{1 \leq i \leq m} \bigwedge_{j \in J_i} x_j > 0 \]

variables over \( \mathbb{Q}_+ \)

coeff. in \( \mathbb{Z} \)
Convex linear Horn constraints

Conjunction of terms of the form

\[ a_1 \cdot x_1 + \ldots + a_n \cdot x_n \sim c \lor \bigvee_{1 \leq i \leq m} \bigwedge_{j \in J_i} x_j > 0 \]

variables over \( \mathbb{Q}_+ \)

coeff. in \( \mathbb{Z} \)

\( \sim \) is \( \geq \) or \( > \)
Convex linear Horn constraints

Examples:

\[ x = 0 \quad \equiv \quad -x \geq 0 \]
Convex linear Horn constraints

Examples:

\[ x = 0 \equiv -x \geq 0 \]

\[ x > 0 \rightarrow \bigvee_{y \in Y} y > 0 \equiv x = 0 \lor \bigvee_{y \in Y} y > 0 \]
Convex linear Horn constraints

Examples:

\[ x = 0 \equiv -x \geq 0 \]

\[ x > 0 \rightarrow \bigvee_{y \in Y} y > 0 \equiv x = 0 \lor \bigvee_{y \in Y} y > 0 \]

\[ x > 0 \leftarrow \bigvee_{y \in Y} y > 0 \equiv \bigwedge_{y \in Y} (y = 0 \lor x > 0) \]
Convex linear Horn constraints

Examples:

\[ x = 0 \equiv -x \geq 0 \]

\[ x > 0 \rightarrow \bigvee_{y \in Y} y > 0 \equiv x = 0 \lor \bigvee_{y \in Y} y > 0 \]

\[ x > 0 \leftarrow \bigvee_{y \in Y} y > 0 \equiv \bigwedge_{y \in Y} (y = 0 \lor x > 0) \]

\[ x > 0 \rightarrow \bigwedge_{y \in Y} y > 0 \equiv x = 0 \lor \bigwedge_{y \in Y} y > 0 \]
Convex linear Horn constraints

Examples:

\[ x = 0 \equiv -x \geq 0 \]

\[ x > 0 \rightarrow \bigvee_{y \in Y} y > 0 \equiv x = 0 \lor \bigvee_{y \in Y} y > 0 \]

\[ x > 0 \leftarrow \bigvee_{y \in Y} y > 0 \equiv \bigwedge_{y \in Y} (y = 0 \lor x > 0) \]

\[ x > 0 \leftarrow \bigwedge_{y \in Y} y > 0 \equiv \text{not expressible} \]
Examples:

- graph reachability
- strong connectivity (from a node)

in subgraph $G[u : x_u > 0]$
Convex linear Horn constraints

<table>
<thead>
<tr>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous Petri net reachability is expressible by a quadratic size formula</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Proof idea</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + C \cdot \pi = y \land \text{firability constraints} )</td>
</tr>
</tbody>
</table>
### Theorem

Satisfiability is PTIME-complete

### Proof idea

- **∈ PTIME:** algorithm exploiting convexity
- **Hardness:** from linear prog. feasibility
Convex linear Horn constraints

Related work:

CSP of linear Horn constraints

(Jonsson and Backström ’98
Koubarakis ’01)
Convex linear Horn constraints

Related work:

CSP of linear Horn constraints

(Jonsson and Backström '98
Koubarakis '01)
Solutions as reachability sets

Theorem

Convex linear Horn constraints solutions are expressible as reachability sets of continuous Petri nets
Solutions as reachability sets

\[ 2x + 3y - 5z > -4 \lor (x > 0 \land y > 0) \lor z > 0 \]

\[ \iff \]

all places can be emptied

all places can be emptied
Continuous VASS:

- Hybrid model
- subsume continuous Petri nets
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$.

Continuous VASS:

- Hybrid model
- Subsume continuous Petri nets

Reachability: $p(u) \not\in Q + q(v)$.

$p(0, 0)$
Continuous VASS

Reachability: $p(u) \not\in Q + q(v)$

Transitions can be scaled by $0 < x \leq 1$.

Continuous VASS:
- Hybrid model
- Subsume continuous Petri nets
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$.

Reachability: $p(u) \not\in Q + q(v)$.

Continuous VASS:
- Hybrid model
- Subsume continuous Petri nets
Reachability: \( p(u) \not\rightarrow Q + q(v) \? \)

Transitions can be scaled by \( 0 < x \leq 1 \).

Continuous VASS:
- Hybrid model
- Subsume continuous Petri nets
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$.

Continuous VASS:
• Hybrid model
• Subsume continuous Petri nets

Reachability:
$p(u) \not\in Q + q(v)$?

$q(1, 0)$

$(1, 0)$

$\frac{1}{2} \cdot (-2, 4)$
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$.

- Hybrid model
- Subsume continuous Petri nets

Reachability: $p(u) \rightarrow Q + q(v)$
Continuous VASS

Transitions can be scaled by $0 < x \leq 1$.

Continuous VASS:

- Hybrid model
- Subsume continuous Petri nets

Reachability: $p(u) \not\in Q + q(v)$.

$p(0, 0) \rightarrow Q_+ q(0, 2)$
Continuous VASS

Reachability: $p(u) \xrightarrow{*} \mathbb{Q}_+ q(v)$?
Theorem

\[ p(u) \xrightarrow{*} q(v) \]

is expressible in \( \exists \text{FO}(\mathbb{Q}, +, <) \)
Theorem

\[ p(u) \xrightarrow{\star} Q_+ q(v) \]

is expressible in \( \exists \text{FO}(Q, +, <) \)
Fixed length runs are easy to express in $\exists\FOL(Q, +, <)$.
Continuous VASS: logical characterization

Theorem $p(u) \not\in Q + q(v)$ is expressible in $\mathcal{EFO}(Q, +, <)$.

Characterization of admissible paths.

Next slide.
Continuous VASS: logical characterization

\[ r(x) \rightarrow \mathbb{Q}+ \quad r(y) \]

Equivalent

\[ r(x) \rightarrow \mathbb{Q} \quad r(y) \]

\[ \land r(x) \rightarrow \mathbb{Q}+ \quad r(\_\_) \]

\[ \land r(\_\_) \rightarrow \mathbb{Q}+ \quad r(y) \]
Characterization of admissible paths

\[
\begin{align*}
  r(x) & \overset{*}{\rightarrow} Q \quad r(y) \\
  \land r(x) & \overset{*}{\rightarrow} Q_+ \quad r(\_)
\end{align*}
\]

\[
\begin{align*}
  \land r(\_) & \overset{*}{\rightarrow} Q_+ \quad r(y)
\end{align*}
\]
Fixed length runs are easy to express in $\mathcal{E} \mathcal{F}O(Q, +, <)$.

Equivalent

Characterization of admissible paths

$\text{Theorem } p(u) \not\leq Q + q(v)$
VASS cyclic $\mathbb{Q}$-reachability

$p(u) \xrightarrow{*} \mathbb{Q} p(v)$?
$p(0, 0) \xrightarrow{\mathbb{Q}} p(6, 0)$?

Diagram:

- Node $p$ transitions to $(0, 0)$ via $(1, 1)^{1/2}$ and $(3, 1)$ via $(1, -1)^{1/2}$.
- $(0, 0)$ transitions to $(1, 1)$ via $(0, 0)$ and $(3, 1)$.
- $(1, 0)$ transitions to $(0, 0)$ via $(0, 0)$ and $(3, 1)$.

VASS cyclic $\mathbb{Q}$-reachability
Guess transitions occurrences
VASS cyclic $\mathbb{Q}$-reachability

Guess scaling factors

![Graph with scaling factors and vectors]
occurrences $\geq$ scaling
occurrences > 0 $\iff$ scaling > 0
\((0, 0) + \sum_{t \in T} f_t \cdot z_t = (6, 0)\)
$G[x_t > 0]$ strongly connected
$G[x_t > 0]$ strongly connected
\[ \sum_{t \in \text{in}(q)} X_t = \sum_{t \in \text{out}(q)} X_t \]
Theorem
These conditions are expressible as convex linear Horn constraints

Corollary
VASS cyclic $\mathbb{Q}$-reachability is in PTIME
Theorem

These conditions are expressible as convex linear Horn constraints

Corollary

VASS cyclic $\mathbb{Q}$-reachability is in PTIME
• new PTIME fragment of $\exists\text{FO}(\mathbb{Q}, +, <)$

• logical characterization of continuous Petri nets

• new hybrid model: continuous VASS

• characterization of $\exists\text{FO}(\mathbb{Q}, +, <)$
Conclusion: future work

• optimization of continuous Petri nets w.r.t. to linear objective functions
  (e.g. constant-rate multi-mode systems of Alur et al. HSCC'12)

• games on continuous Petri nets
  (e.g. Minkowski games of Le Roux et al. STACS'17)

• more succinct encoding of continuous VASS reachability
Thank you!
Takk fyrir!