The marking equation

Let \( \sigma \) be a long sequence

\[
\sigma = t_1 t_2 t_3 t_4 t_5 t_6 t_7 \rightarrow \ \text{long suffix} \quad X = (2, 3, 3, 0)
\]

\[
X' = (2, 3, 2, 2, 6, 4, 0)
\]

\[
M(p) = 1 + 2.1 + 3.2 - 2.2 - 2.1 - 6.0 + 1.0 + 0.0 = 3
\]

\[
M(p) = M_0(p) + X(h) \left( W(h_1, p) - W(p, h_1) \right) \quad \text{I. Term}
\]

\[
+ X(h) \left( W(h_1, p) - U(p, h_1) \right) \quad \text{II. Term}
\]

\[
+ X(h) \left( W(h_1, p) - U(p, h_1) \right) \quad \text{III. Term}
\]

\[
M(p) = M_0(p) + \sum_{i=1}^{n} X(t_i) \left( \underbrace{W(t_i, p) - U(p, t_i)}_{\text{Correlation}} \right)
\]

\[
M(p) = M_0(p) + \left( C \cdot X \right) (p)
\]

\[
M = M_0 + C \cdot X \quad \text{Marking equation}
\]
The working equation of a cyclic net

**Theorem** Let $N = (P_i, E, W, M_0)$ be a cyclic net and let $M$ be a member of $N$. $M$ is reachable \( \iff \) there exists $X \in N^{+}$ such that

$$M = M_0 + C \cdot X$$

where $C$ is the incidence matrix of $N$.

**Proof** \( \Rightarrow \) If $M_0 \xrightarrow{0} M$, then there exists $X$ such that $X \neq 0$.

\( \Leftarrow \) By induction in $K = \sum_{t \in T} X(t)$

*Base* $K = 0$. Then $X = (0, 0, \ldots, 0)$.

*Step* $K > 0$. Say $t_1 \leq t_2$. If there is a (possibly empty) path from $t_1$ to $t_2$. Since $N$ is cyclic, $\leq$ is a partial order.

Let $\|X\|$ be the set of transitions $t$ such that $X(t) > 0$.

Let $t_m$ be a minimal element of $\|X\|$ wrt $\leq$.

We prove that $M_0$ reaches $t_m$.

![Diagram](attachment:image.png)

If $M_0$ does not reach $t_m$, then there is $p \in t_m$ such that $M_0(p) = 0$. Since $X(t) = 0$ for every $t \in p \setminus t_m$ and $X(t_m) > 0$, we have $M(p) = M_0(p) + (C \cdot X)(p) < 0$, contradicting the fact that $M$ is a member.
So no cycle, \( t \in \mathbb{N} \). Let \( M_0 \xrightarrow{t \in \mathbb{N}} M_1 \).

Consider the input count vector \( X' \):

\[
X'(t) = \begin{cases} 
X(t) + 1 & \text{if } t = t_0 \\
X(t) & \text{otherwise}
\end{cases}
\]

We have \( M = M_1 + C \cdot X' \):

\[
M = M_0 + C \cdot X = M_0 + C \cdot (X + e_1 + e_{t_0}) = M_1 + C \cdot X' = M_1 + C \cdot X' \]

Since \( \sum_{t} X'(t) < \sum_{t} X(t) \) by induction hypothesis, there exists \( t^* \) with \( M_1 \xrightarrow{t^*} M \).

But then \( M_0 \xrightarrow{t \in \mathbb{N}} M_1 \xrightarrow{t^*} M \) and \( M \) is reachable.
Theorem: The reachability problem for acyclic Petri nets
is NP-complete.

Proof: Reachability is in NP

Checking if \( M = M_0 + C \cdot X \) has some solution \( X \in \mathbb{N} \)
is in NP

Reachability is NP-hard.

By reduction from CNF-SAT

\[ \phi = (x_1 \lor \neg x_2) \land (x_1 \lor x_2) \]

Reachability of \( \phi \) satisfying \( \phi \) with exactly one true
in \( C_1 + C_0 \) and 0 tokens elsewhere is
reachable.

Exercise: Show NP-hardness of

Given: a Petri net \( N \), a place \( p \) \in \( N \)

Decide: is there a reachable marking \( M \) such that
\( M(p) = 1 \)?