Definition. A set \( M \) of markings is upward-closed if

\[ M \subseteq M \text{ and } M' \supseteq M \implies M' \in M \]

A marking \( M \) of an upward-closed set \( M \) is maximal if there is no \( M' \in M \), \( M' \supseteq M \) such that \( M' \neq M \).

Lemma. Every upward-closed set has finitely many maximal elements.

Proof. Assume the contrary. Then there exists an infinite sequence of maximal markings, pairwise different. By Dickson’s Lemma, there are \( i < j \) such that \( M_i \leq M_j \). But then \( M_j \) is not maximal.

Definition. Let \( M \) be a set of markings. Let \( t \) be a transition.

We define:

\[ \text{pre}(M, t) = \{ M' \mid M' \not\rightarrow M \text{ for some } M \in M \} \]

\[ \text{pre}(M) = \bigcup_{t \in T} \text{pre}(M, t) \]

Lemma. If \( M \) is upward-closed, then \( \text{pre}(M) \) is also upward-closed.

Proof. \[ \text{pre}(M) \in M \]

\[ \text{pre}(M) \not
to \]

\[ \text{pre}(M) \in M \]

Definition. Define

\[ \text{pre}^0(M) = M \]

\[ \text{pre}^i(M) = \text{pre}(\text{pre}^{i-1}(M)) \]

\[ \text{pre}^*(M) = \bigcup_{i=0}^{\infty} \text{pre}^i(M) \]
Theorem. There is $i > 0$ such that $\text{pre}^*(M) = \bigcup_{j=0}^i \text{pre}^j(M)$.

Proof. a) $\text{pre}^*(M)$ is upward-closed.

Because union of upward-closed sets is upward-closed.

Let $M^*$ be the set of minimal elements of $\text{pre}^*(M)$.

We know that $M^*$ is finite.

Let $M_i$ be the set of minimal elements of $\text{pre}^i(M)$.

Let $i$ be the smallest index such that $M^* \subseteq \bigcup_{j=0}^i M_j$.

We then have $\text{pre}^*(M) = \bigcup_{j=0}^i \text{pre}^j(M)$.

Good reading $M$

$M := \{ M^* \mid M^* \supseteq M \}$

$\text{Old } M := \emptyset$

while $M \neq \text{Old } M$ and $M_0 \notin M$

$\text{Old } M := \{ M_0 \}$

$M := M \cup \text{pre}^0(M)$

$\text{pre}^0(M)$

if $M_0 \in M$ then continue "good"

else continue "not good"

$\min \left( \text{pre}(M) \right) = \min \left( \text{pre}(\min(M)) \right)$

$\min \left( M_i, M_j \right) = \min ( \min (M_i), \min (M_j) )$