Exercise 1

(a) Recall the definition of the if-then-else operator \( \text{ite} \):
\[
\text{ite}(F, G, H) \equiv (F \land G) \lor (\neg F \land H).
\]
Show how to express \( A \leftrightarrow \neg B \) using only \( \text{ite} \), \( A \), \( B \), and the constants 0 and 1 (representing false and true, respectively).

(b) W.r.t. the variable order \( x < y < r < c \) construct the BDD representing the following formula:
\[
F = (r \leftrightarrow (x \leftrightarrow \neg y)) \land (c \leftrightarrow (x \land y)).
\]

Exercise 2

Skolemize the following formula. In every step, state how the formula was transformed and whether semantic equivalence or only equisatisfiability holds:
\[
F = \neg \exists x \forall y (P(x, y) \land \exists z (P(x, z) \lor Q(z))).
\]

Exercise 3

Consider the following formulas where \( a, b \) are constants, and \( P, Eq \) are predicate symbols:
\[
F_1 = \forall x \forall y \forall z \forall v \left( (P(x, y, z) \land P(x, y, v)) \rightarrow Eq(z, v) \right)
\]
\[
F_2 = \forall x \left( P(x, a, x) \land P(b, x, x) \right)
\]
\[
F_3 = Eq(b, a)
\]
Show that \( G = (F_1 \land F_2) \rightarrow F_3 \) is valid using resolution.

Remark: State clearly intermediate results so that if you make a mistake you do not lose all points.

Exercise 4

(a) Assume \( F \) is a satisfiable formula of first-order logic in clause form with an infinite Herbrand universe.
Is it true that every model of \( F \) has an infinite universe? Prove your answer correct.

(b) Give an example of a satisfiable formula \( F \) (w/o equality!, not necessarily in clause form) such that every model of \( F \) has an infinite universe.
Remark: You do not have to prove that your formula has the required property.

(c) Skolemize the formula
\[
F = \exists x P(x) \lor \exists y P(y) \lor \forall z P(z)
\]
in three different ways yielding formulas \( G_1, G_2, G_3 \) such that for the Herbrand universe \( D(G_i) \) it holds that
i) \( D(G_1) \) consists of exactly one element,
ii) \( D(G_2) \) consists of exactly two elements, and
iii) \( D(G_3) \) is infinite.
Exercise 5

The semantics of the uniqueness quantifier $\exists!x$ (read: there exists a unique $x$ such that . . . ) is defined as follows:

$$
A \models \exists!xF \text{ if and only if } \text{there exists } d_0 \in U_A \text{ such that } A_{[x:=d_0]} \models F
$$

and for all $d \in U_A$ if $A_{[x:=d]} \models F$, then $d = d_0$.

Prove each of the nonequivalences stated below: That is, for each nonequivalence $Qx\exists!yF \not\equiv \exists!yQxF$ give a formula $F$ and a structure $A$ so that $A$ is suitable for both formulas $Qx\exists!yF$ and $\exists!yQxF$, but only a model for one of them.

(a) $\forall x\exists!yF \not\equiv \exists!y\forall xF$  
(b) $\exists x\exists!yF \not\equiv \exists!y\exists xF$  
(c) $\exists!x\exists!yF \not\equiv \exists!y\exists!xF$.

Remark: Try to interpret the formulas as statements on directed graphs.

Exercise 6

For each of the following sets $L$ of literals compute (from left to right, as in the algorithm discussed in the lecture) a most general unificator $\text{sub}$ and the result $L_{\text{sub}}$ of the unification if $\text{sub}$ exists; otherwise state why $\text{sub}$ does not exists.

(a) $L = \{P( g(f(x_1), x_2), f(g(x_1, x_2)) ), P( g(y_1, f(y_2)), f(g(y_3, y_4)) ) \}$.

(b) $L = \{P( g(f(x_1), f(x_2)), f(g(x_3, x_2)) ), P( g(y_1, f(y_2)), f(g(y_3, f(y_2))) ) \}$.

(c) $L = \{P( g(f(x_1), x_2), f(g(x_1, x_2)) ), P( g(y_1, y_3), f(y_5) ), P( g(y_1, f(y_2)), f(g(y_3, y_4)) ) \}$.

(d) $L = \{\neg P( g(f(x_1), x_2), f(g(x_1, x_2)) ), P( g(y_1, y_3), f(y_5) ) \}$.

Exercise 7

Let $T_1 \subseteq T_2$ be two theories of first-order logic.

Prove or refute each of the following statements:

(a) If $T_2$ is decidable, then so is $T_1$.

(b) If $T_2$ is complete, then so is $T_1$.

(c) If $T_2$ is consistent, then so is $T_1$.

Remark: Only yes/no does not suffice, you have to explain why the statement holds or does not hold.

Exercise 8

Given a set $\mathcal{X}$ of propositional formulas, let $\text{Cn}(\mathcal{X})$ denote the set of consequences of $\mathcal{X}$, i.e., the set of all propositional formulas $F$ with $\mathcal{X} \models F$.

Let $\mathcal{X}$ be an arbitrary set of propositional formulas, and $\mathcal{Y} = \{F_1, \ldots, F_n\}$ a finite set of propositional formulas (not necessarily included in $\mathcal{X}$) such that $\text{Cn}(\mathcal{X}) = \text{Cn}(\mathcal{Y})$.

(a) Prove that there is a finite subset $\mathcal{X}' \subseteq \mathcal{X}$ such that $\text{Cn}(\mathcal{X}') = \text{Cn}(\mathcal{Y})$ using the compactness theorem.

(b) Give an alternative proof for the result of (a) but this time based on the results regarding the Hilbert calculus.