TOP-DOWN SYNTHESIS OF LIVE AND BOUNDED FREE CHOICE NETS

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ABSTRACT: The paper provides a set of rules for the stepwise synthesis of all and only live and bounded Free Choice nets. The starting point are nets composed by a circuit containing one place and one transition.

KEYWORDS: Free Choice nets, synthesis, structure theory.

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1. INTRODUCTION
Petri Nets are an effective tool for modelling concurrent systems, which combine an appealing graphical representation with a mathematical formalism. This formalism can be used to obtain synthesis rules which, starting from a simple system, permit to modify and refine it while preserving properties of good behaviour such as, typically, liveness and boundedness. A lot of effort has been done in this direction. The work of Berthelot [BERT 83, BERT 87], Suzuki and Murata [SUMU 83] or Valette [VALE 79] has produced synthesis techniques which allow to build impressive examples. Nevertheless, a problem of this approach is that no characterisation of the expressive power of the techniques is given.

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A possible solution is the determination of which rules should be used to obtain all and only the well-behaved models of a certain class. This option has been followed by Genrich and Thiagarajan in [GETH 84]. This pioneer and beautiful work has been our main source of inspiration from a conceptual point of view. Bipolar Schemes are models of computation closely related to Petri Nets, which allow to model both concurrency and nondeterminism. The authors provide a set of rules for the synthesis of all and only well-behaved Bipolar Schemes, together with a mechanism to translate them into Live and Safe Free Choice nets (LSFC nets). Unfortunately, well-behaved Bipolar Schemes correspond to only a proper subset of all LSFC nets. At the end of [THVO 84], Thiagarajan and Voss expressed their feeling that LSFC nets would eventually admit a synthesis theory. If we replace "safe" by "bounded", this is the aim of the present work. Incidentally, they also expressed their confidence that the results of [THVO 84] would play an useful role in the development of such a theory, and certainly that has been the case. We develop here a top-down synthesis theory, along the line of Bipolar schemes. We understand as top-down synthesis a procedure that leads to the final model through stepwise transformations.

Very recently, two papers by Desel [DESE 90] and Kovalyov [KOVA 91] have provided sets of reduction rules which reduce some subclasses of live and bounded Free Choice nets to particularly simple ones. The "inverses" of these rules permit to synthesise all and only the nets in these subclasses. For a more detailed comparison of our work with the results in these two papers the reader is referred to the conclusions.

The paper is organised as follows. Section 2 introduces basic concepts and results. In section 3 the results of Free Choice structure theory that we require are stated (it is interesting to notice that they were produced along 18 years). We also make use of some known and some new properties and concepts on State Machine Decomposable nets (SMD nets). They are considered in section 4. In section 5 the rules of our synthesis procedure are introduced. Section 6 presents the synthesis procedure, together with an example of application. Section 7 (the most technical section) proves the completeness of the procedure.

2. BASIC DEFINITIONS AND RESULTS

The model we consider in this paper are Place/Transition nets all whose arcs have weight 1 (also called ordinary P/T-nets). We assume that the reader is familiar with this model, as well as with the notions of liveness, boundedness and incidence matrix. Less standard concepts and notations are defined below.

Definition 2.1 Let <N, M_o> be a (P/T-) marked net. R(N, M_o) is the set of all markings reachable from M_o. L(N, M_o) is the set of all firing sequences applicable from M_o: L(N, M_o)={σ | M_o[σ>M].

Definition 2.2 N is structurally bounded (SB for short) iff for every initial marking M_o: <N, M_o> is bounded. N is structurally live (SL for short) iff there exists an initial marking M_o such that <N, M_o> is live.

Definition 2.3 (subnets). Let N=(P, T; F) and N'=(P', T'; F') be two nets. N' is a partial subnet of N (denoted by N'⊆N) iff P'⊆P, T'⊆T and F'⊆F. N' is a subnet of N (denoted by N'⊆N) iff P'⊆P, T'⊆T and F'=F∩((P'×T')∪(T'×P')). A place p∈P' is a way-in (way-out) place of N' iff there exists tε p (tε p*) such that tε T' (the dot notation refers to F). The transition t is called a source (a sink) of N' (see figure 1).

Definition 2.4 (classes of nets). Let N=(P, T; F) be a net. N is a P-graph or State Machine iff ∀tε T: l^t=0=l^t=1. N is a T-graph or Marked Graph iff ∀pε P: l^p=0=l^p=1. N is Free Choice iff ∀pε P such that l^p>1: (p*)=(p).
N is a P-component (T-component) of N iff N' is a strongly connected P-graph (T-graph) and satisfies \( T'=P' \cup T' \) (\( P'=T' \cup T' \)). N is State Machine Decomposable iff there exists a set \( R=\{N_1, ..., N_k\} \) of P-components of N with \( N_i=(P_i, T_i; F_i) \), such that \( P=\cup P_i \), \( T=\cup T_i \), \( F=\cup F_i \). \( R \) is called a cover by P-components (cover, for short) of N, and it is said that N is covered by \( R \). N is Marked Graph Decomposable iff there exists a cover by T-components of N. A cover \( R \) is minimal iff none of its proper subsets is also a cover.

P-components are denoted with subscripts and T-components with superscripts.

**Definition 2.5** (paths and circuits). A path of a net N is a sequence \( \Pi=(x_1, x_2, ..., x_r) \), \( r \geq 2 \), of elements of \( P \cup T \) such that \( \forall i \ 1 \leq i \leq r-1 \): \( (x_i, x_{i+1}) \in F \). \( \Pi \) is elementary iff all \( x_i \) are distinct, except possibly \( x_1 \) and \( x_r \). A subsequence \( (x_i, x_{i+1}, ..., x_{i+k}) \) is a subpath of \( \Pi \). Let \( \Pi_1=(x_1, ..., x_q) \) and \( \Pi_2=(y_1, ..., y_r) \) be paths such that \( x_q=y_1 \). The path \( \Pi=(x_1, ..., x_q=y_1, ..., y_r) \) is called the concatenation of \( \Pi_1 \) and \( \Pi_2 \) (denoted by \( \Pi=\Pi_1;\Pi_2 \)). Let \( x_i \) be the first node of \( \Pi_1 \) such that \( x_i \in \Pi_2 \). Consider the two subpaths \( \Pi_1'=(x_1, ..., x_i) \) and \( \Pi_2'=(x_i, ..., y_r) \) of \( \Pi_1 \) and \( \Pi_2 \) respectively. If \( \Pi_1 \) and \( \Pi_2 \) are elementary, then \( \Pi'=\Pi_1';\Pi_2' \) is elementary as well. In this case \( \Pi' \) is called the elementary concatenation of \( \Pi_1 \) and \( \Pi_2 \) (denoted by \( \Pi'=\Pi_1;;\Pi_2 \)). A general circuit of N is a path \( \Gamma=(x_1, ..., x_r) \) such that \( x_1=x_r \). A general circuit is elementary (or just a circuit, for short) iff it is elementary as a path.

The following definition presents handles and bridges, the structural objects introduced in [ESSI 89] for the study of Free Choice nets.

**Figure 1.** A net N (on the left) and one of its subnets N' (on the right). The way-in places of N' are \( p_1, p_2 \). The way-out places are \( p_2, p_3 \) (notice that a place can be at the same time way-in and way-out). The sources of N' are \( t_1, t_2 \). The sinks are \( t_3, t_4, t_5 \).

**Definition 2.6** (handles, bridges, forks, branches). Let \( N'=(P', T'; F') \), \( N''=(P'', T''; F'') \) be partial subnets of N. An elementary path \( (x_1, ..., x_r) \) is a handle of \( N' \) iff \( \{x_1, ..., x_r\} \cap (P' \cup T')=\{x_1, x_r\} \). \( (x_1, ..., x_r) \) is a bridge from \( N' \) to \( N'' \) iff \( \{x_1, ..., x_r\} \cap (P' \cup T'')=\{x_1\} \) and \( \{x_1, ..., x_r\} \cap (P'' \cup T'')=\{x_r\} \).
A place \( p \in P' \) is a \textit{branch} of \( N' \) iff there is \( \tau \in p^* \) such that \( \tau \in T' \). The path \((p, t)\) is called a \textit{branching path}. A transition \( \tau \in T' \) is a \textit{fork} of \( N' \) iff there is \( \tau \in p^* \) such that \( \tau \in P' \). The path \((t, p)\) is called a \textit{forking path}. It is said that a branching or forking path \((x, y)\) can be \textit{extended to a handle} iff there exists a handle \( H = (x, y, ..., z) \) of \( N' \).

Figure 2 shows an example of a handle and a bridge. Handles and bridges are classified according to the nature of their first and last nodes. The four types are denoted PP-, PT-, TP- and TT-handles or bridges.

Some propositions in the paper are proven by showing that a partial subnet \( N' \) has a handle of a certain type. The handle is obtained taking a node \( x \) that is not in \( N' \) and proving the existence of two bridges, the first from \( N' \) to \( x \) (in fact, to the net consisting of the node \( x \) and no arcs, but we allow us here an abuse of language) and the second from \( x \) to \( N' \). The elementary composition of these two bridges (which is an elementary path) yields the desired handle.

\[ \begin{array}{c}
\text{2a. A TP-handle of a circuit} \\
\text{2b. A PT-bridge between two paths}
\end{array} \]

\textbf{Figure 2. A handle and a bridge.}

**Definition 2.7 (semiflows and conservativity).** Let \( C \) be the incidence matrix of \( N = (P, T; F) \). An integer valued vector \( Y \) of dimension \( |P| \) is a \textit{P-semiflow} of \( N \) iff \( Y^T.C = 0 \) and \( Y \geq 0 \), \( Y \neq 0 \). The set \( \|Y\| = \{p \in P \mid Y(p) > 0\} \) is the \textit{support} of \( Y \).

\( N \) is \textit{conservative} iff there exists a rational valued vector \( Y > 0 \) such that \( Y^T.C = 0 \).

\textbf{Proposition 2.1 [MERO 80].} If \( N \) is conservative, then it is structurally bounded. If \( N \) is SL&SB, then it is conservative.

\section{Some Results on Free Choice Nets.}

In this section we introduce some results of structure theory of Free Choice nets that we have employed to work out the synthesis procedure and prove it sound and complete. The main one is theorem 3.1, which characterizes liveness and boundedness. It is an easy-to-prove modification of the characterisation of SL&SB given in [ESCI 89].

\textbf{Theorem 3.1 [ESPA 90].} Let \( <N, M_0> \) be a strongly connected Free Choice net. \( <N, M_0> \) is live and bounded iff for every strongly connected P-graph \( N' = (P', T', F') \leq N \) the following three conditions hold:

\begin{enumerate}
\item No forking path of \( N' \) can be extended to a TP-handle.
\item Every branching path of \( N' \) can be extended to a PP-handle.
\item If \( N' \) is a P-component of \( N \), then \( P' \) is marked at \( M_0 \) (i.e. at least one place of \( P' \) has a token at \( M_0 \)).
\end{enumerate}

In particular, circuits are strongly connected P-graphs, and therefore conditions (a) and (b) apply to them. Notice also that this theorem has an important consequence: if \( N \) is a Free
Choice net that admits live and bounded markings, then these markings are exactly those satisfying condition (c).

Theorem 3.2 [HACK 72] [BETH 87] (decomposability theorems). Let \(<N, M_0>\) be a live and bounded Free Choice net. \(N\) is State Machine Decomposable and Marked Graph Decomposable.

Live and bounded Free Choice is shortened in the sequel to LBFC. The net of Figure 3 is LBFC. It is covered by the P-components with sets of places \(\{p_1, p_2, p_4, p_6\}\) and \(\{p_1, p_3, p_5, p_7\}\) respectively. It is also covered by the T-components with sets of transitions \(\{t_1, t_3, t_4, t_7\}\) and \(\{t_2, t_5, t_6, t_7\}\).

![Figure 3. Marked net used to illustrate several theorems of section 3.](image)

Theorem 3.3 [THVO 84] (maximal P-graphs are P-components). Let \(<N, M_0>\) be an LBFC net and \(N\)' a strongly connected P-graph such that \(N\)'\(\subseteq\)\(N\). \(N\)' is a P-component of \(N\) iff there is no strongly connected P-graph \(N''\) such that \(N''\subseteq\)\(N\). The same statement holds replacing P-graph by T-graph and P-component by T-component.

In particular, since a circuit is both a strongly connected P-graph and a strongly connected T-graph, every circuit of a LBFC net is contained in some P-component and in some T-component.

The following result is a direct consequence of theorem 3.1, which has been proved in several papers, always as a corollary of different characterisations of liveness and boundedness. It states that adding tokens to an LBFC net we cannot kill it.

Theorem 3.4 (liveness monotonicity). Let \(<N, M_0>\) be an LBFC net and \(M_0'\geq M_0\). Then \(<N, M_0'>\) is also live and bounded.
4. SOME DEFINITIONS AND RESULTS ON STATE MACHINE DECOMPOSABLE NETS.

Hack's decomposability theorem shows that every LBFC net is also SMD. In order to prove that our synthesis procedure is complete, we profit from this fact by making use of some properties of this subclass. They are introduced now together with some terminology.

**Proposition 4.1.** (i) SMD nets are conservative and structurally bounded.
   (ii) LBFC nets are conservative and SL&S

**Proof.** (i) Use the well known fact that the set of places of a P-component is the support of a P-semiflow. (ii) LBFC nets are SMD by theorem 3.2. Apply then (i) ♦

In the next proposition we deal with the subnets covered by a subset of the P-components of a certain cover. The intersection of a T-component of the big net with one of these subnets turns out to be a set of T-components of the subnet. In other words, T-components "distribute" over P-components.

We need a notation for the pre- and post sets of elements of a net N' that is a subnet of a net N. *x denotes the pre-set of x in N, and (x)N the corresponding pre-set in N'.

**Lemma 4.1.** Let N1=(P1, T1; F1), N1=(P1, T1; F1) be a P-component and a T-component of a net N respectively. Then N'=N1∩N1 (defined componentwise) is a (possibly empty) set of pairwise disjoint circuits of N.

**Proof.** If N'≠∅, we are done. Assume therefore that N'≠∅. Take x∈P∪T. Suppose x is a place. Then, as x∈P1 and N1 is a P-component of N, x∩x∩T1=1. As N1 is a T-component of N, l(x)∩T1=1 and l(x)∩T1=1. Therefore l(x)N'=l(x)N=1. If x were a transition, the same result would be obtained interchanging the roles of N1 and N1. Therefore every node of N' has exactly one input node and one output node. It is easy to see that the only nets that satisfy this condition are the ones formed by a set of pairwise disjoint elementary circuits. ♦

Figure 4 shows that this set of circuits can contain more than one element.

The next proposition is a generalisation of the previous lemma, in which instead of P-components we consider subsets of P-components.

**Proposition 4.2.** Let N be a SMD net and R a cover of N. Let N'=(P', T'; F') be the subnet of N covered by R'⊆R, and N1=(P1, T1; F1) a T-component of N. Then N''=(P'', T'', F'')=N'∩N1 is a (possibly empty) set of pairwise disjoint T-components of N'.

**Proof.** If N''=∅, we are done. Assume therefore that N''≠∅. We make the following three claims:

1. ∀p∈P'': |(o)p|N'' ≤ 1 and |(p')|N'' ≤ 1.
   **Proof of claim 1.** Obvious, because N1 is a Marked Graph.

2. ∀t∈T'':("t")N''∩("t")N''⊆P''
   **Proof of claim 2.** Since N1 is a T-component of N, *(t)N∪*(t)N⊆P1 and ("t")N∪("t")N⊆P'. Therefore ("t")N∩("t")N⊆P'∩P1=P''.


Every connected component of $N''$ is strongly connected.

**Proof of claim 3.** $N' = \bigcup N_i'$, where $N_i'$ are the $P$-components of $R$. Then $N'' = N' \cap N^1 = (\bigcup N_i') \cap N^1 = \bigcup (N_i' \cap N^1)$. Using now lemma 4.1, each $N_i' \cap N^1$ can be covered by elementary circuits, and therefore the same holds for $N''$. This implies that every connected component of $N''$ is strongly connected.

Claims (1) and (3) imply that each of the connected components of $N''$ is a strongly connected T-graph. The addition of (2) proves the desired result.

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Figure 4 is also an example of this theorem, since the net shown there is SMD. It is easy to check, as proposition 4.2 ensures, that the circuits of the intersection are T-components of the P-component marked with shaded places in the upper left net of the figure.

Let us introduce now some vocabulary about SMD nets. The first two notions are those of *environment* of a P-component, and *rendez-vous* of a P-component with the environment. Rendez-vous can be considered as the interface between the P-component and the rest of the system.
**Definition 4.1.** Let \( N=(P, T; F) \) be a SMD net, \( \mathcal{R} \) a cover of \( N \) and \( N_i=(P_i, T_i; F_i) \in \mathcal{R} \). The subnet \( \overline{N}_i=(\overline{P}_i, \overline{T}_i; \overline{F}_i) \) covered by \( \mathcal{R} \setminus N_i \) is called the *environment* of \( N_i \). A transition \( t \in T_i \) is a *rendez-vous* of \( N_i \) iff \( t \in \overline{T}_i \setminus T_i \).

We give next a definition corresponding to the idea of internal behaviour of a P-component, i.e. to those actions that the P-component can perform freely, without being conditioned or disturbed by the environment. This is the idea lying behind the notion of *maximal private* subnet of a P-component.

**Definition 4.2.** Let \( N_i \) be a P-component of a net \( N \) and \( N_i'=(P_i', T_i'; F_i') \) a subnet of \( N_i \). \( N_i' \) is *private* to \( N_i \) iff \( N_i' \cap \overline{N}_i = \emptyset \). Consider the set \( PC(N_i) \) of private connected subnets of \( N_i \). \( N_{ij} \in PC(N_i) \) is *maximal* iff there exists no \( N_{ik} \in PC(N_i) \), \( k \neq j \), such that \( N_{ij} \subseteq N_{ik} \). The set of maximal private connected subnets (or MPC subnets for short) of \( N_i \) is denoted by \( MPC(N_i) \).

Some facts about MPC subnets can be easily proved:

1. If \( \mathcal{R} \) is minimal, then \( MPC(N_i) \neq \emptyset \) (i.e. every P-component of a minimal cover has some private part).
2. If \( N \) is a P-graph and \( \mathcal{R} \) is minimal then \( \mathcal{R} = \{N\} \) and \( MPC(N) = \{N\} \).
3. If \( \mathcal{R} \) is minimal and contains more than one element, then every MPC subnet has at least one way-in place (and therefore at least one source) and at least one way-out place (and therefore at least one sink).
4. Every place \( p \) of an MPC subnet \( N_i'=(P_i', T_i'; F_i') \) satisfies: \( p \) is connected to at least one way-out place by an \( F_i' \)-path, and there exists at least one way-in place connected to \( p \) by an \( F_i' \)-path.

Since the net of figure 3 is SMD, we can use it as an example of these properties. Notice that the cover \( \mathcal{R} = \{N_1, N_2\} \) shown there is minimal. Figure 5 shows the MPC subnets of this cover.

![Diagram](image)

**Figure 5.** MPC subnets of the net of figure 3 with respect to the cover shown there, \( \mathcal{R} = \{N_1, N_2\} \). \( N_1' \) is the only MPC subnet of \( N_1 \), and \( N_2' \) the only MPC subnet of \( N_2 \).

Notice that way-out places of MPC subnets in SMD Free Choice nets have one and only one output transition, which is one of the sinks of the MPC subnet. This follows easily from the Free Choice property. In the net of figure 3, whose MPCs are shown in figure 5, \( p_6, p_7 \) are the way-out places of \( N_1' \) and \( N_2' \) respectively. Both have exactly one output transition, \( t_7 \) in figure 3.
5. THE TRANSFORMATION RULES

In this section we introduce the two rules that compose our synthesis procedure. We call them synthesis rules. We introduce their reverse as well, called reduction rules (if we transform a net using a rule, we can recover the original net by means of the reverse rule). When we refer to both of them, we speak about transformation rules. The reduction rules are used to prove the synthesis procedure complete: the general idea of the completeness proof is that if every LBFC net can be reduced to an atomic one (defined below) by means of a sequence of reduction rules, the reverse sequence of the corresponding synthesis rules synthesises the net.

The obtention of the reverse of a given reduction/synthesis rule is here not always completely straightforward, but also not a difficult task. Therefore the rule and its reverse are presented together. The format of the rules is similar to the given in [GETH 84]. A transformation rule transforms a source marked net \(<N, M_0>\) into a target marked net \(<\bar{N}, \bar{M}_0>\) (in a synthesis rule \(\bar{N}\) is bigger than \(N\), while the converse happens for reduction rules), and its specification consists of four parts:

- **Structural conditions**: the restrictions that \(N\) must satisfy in order for the rule to become applicable.
- **Marking conditions**: the same for \(M_0\).
- **Structural changes**: the change effected in \(N\) to yield \(\bar{N}\).
- **Marking changes**: the specification of \(\bar{M}_0\), the marking of \(\bar{N}\).

A rule is said to preserve a given property when the property holds for the target net \(\text{iff}\) it also holds for the source net.

5.1 The macroplace rules

Certain subnets can be substituted by one place, called macroplace, while preserving liveness and boundedness (in fact, the actual bound of the net is preserved as well).

**Definition 5.1** \(N'=(P', T'; F')\) is reducible to a place if:

(a) \(N'\) is a P-graph containing at least one transition and \(\forall t \in T: l^* \cap P^t \leq 1\) and \(l'^* \cap P' \leq 1\)

(b) For every \(p' \in P'\): there exists at least an \(F'\)-path from a way-in place of \(N'\) to \(p'\).

(c) For every \(p' \in P'\) and every way-out place \(p_0'\) of \(N'\), there exists an \(F'\)-path from \(p'\) to \(p_0'\).

The subnet \(N'\) of figure 1 is not reducible to a place, since it does not satisfy (c) (\(p_2\) is a way-out place and there is no path from \(p_3\) to \(p_2\)). On the other hand, both subnets \(N_1'\) and \(N_2'\) (figure 5) of the net of figure 3 are reducible to a place.

**Remark 5.1.** If \(N\) is a strongly connected P-graph containing more than one transition, then \(N\) contains a subnet reducible to a place.

**Definition 5.2** Let \(<N, M_0>\) be a marked net with \(N=(P, T; F)\) and \(N'=(P', T'; F')\subseteq N\) reducible to a place. The net \(N_r=(P_r, T_r; F_r)\), where

- \(P_r=(PP' \cup MP)\)
- \(T_r=(TT')\)
- \(F_r=(FR \cup (P_rT_r \cup (T_rP_r))) \cup F_{MP}\), and \(F_{MP}\) are the input and output arcs of \(MP\) given by:
  - \((t, MP) \in F_{MP}\) iff there exists \((t, p) \in F\) with \(p \in P'\)
  - \((MP, t) \in F_{MP}\) iff there exists \((p, t) \in F\) with \(p \in P'\)

is a macroplace reduction of \(N\), and \(MP\) is the macroplace that replaces \(N'\).

The marked net \(<N_r, M_r>\), where \(N_r\) is the macroplace reduction of \(N\) and \(M_r\) is given by:

- \(M_r(p)=M_0(p)\) if \(p \neq MP\)
- \(M_r(MP)=M_0(P')\) (the total number of tokens in the places of \(P'\))

is called a macroplace reduction of \(<N, M_0>\).
Figure 6 shows the net obtained after reducing both $N_1'$ and $N_2'$ in the net of figure 3 to the corresponding macroplaces.

![Net diagram]

**Figure 6.** Net obtained from the net of figure 3 by reducing $N_1'$ and $N_2'$ to the macroplaces $MP_1$ and $MP_2$ respectively.

The usefulness of the definition of macroplace lies in the following theorem.

**Theorem 5.1** [SILV 81]. Let $<N, M_0>$ be a marked net and $N' \subseteq N$ reducible to a place. Then $<N_r, M_r>$ is live iff $<N, M_0>$ is live, and $<N_r, M_r>$ is $k$-bounded iff $<N, M_0>$ is $k$-bounded.

It is now easy to derive from theorem 5.1 a reduction and a synthesis rule preserving liveness and boundedness (in the case of boundedness, the actual bound of the net is also preserved).

**Reduction Rule R1**

*Structural conditions:* $N$ contains a subnet $N'$ reducible to a place.

*Marking conditions:* none.

*Structural changes:* $\bar{N} = N_r$, where $N_r$ is the macroplace reduction of $N$ obtained by reducing $N'$ to a place.

*Marking changes:* $\bar{M}_0 = M_r$, where $M_r$ is given by definition 5.2.

**Synthesis Rule S1**

*Structural and marking conditions:* none

*Structural changes and marking changes:* $<\bar{N}, \bar{M}_0>$ is such that $<N, M_0>$ is a macroplace reduction of it.

Notice that $S1$ refines a place by a P-graph. Intuitively, the degree of concurrency of the structure does not change. In order to have a complete synthesis procedure we need at least one more rule able to add concurrency. This rule is given in the next section.

### 5.2 The marking structurally implicit place rules

Along this section $N_p$ denotes the net obtained by adding a place $p$ to a net $N$. The incidence matrices of $N$ and $N_p$ are called $C$ and $C_p$ respectively. The row corresponding to the place $p$ in $C_p$ is called $l_p$ (i.e. $C_p$ can be obtained adding the row $l_p$ to $C$). If $M$ is a marking of $N$, $M \cup m(p)$ denotes the marking of $N_p$ whose restriction to $N$ is $M$ and puts $m(p)$ tokens in $p$.

With this notation conventions we can already define marking structurally implicit places.
**Definition 5.2** The place \( p \) is a **marking structurally implicit place** (MSIP) in \( N_p \) iff there exists a rational-valued vector \( Y \geq 0 \) such that \( Y^T.C = 1_p \).

Loosely speaking (if we identify a place with its corresponding row in the incidence matrix), MSIPs are linear combinations of other places with nonnegative coefficients. The reason of the name MSIP lies in the following property.

**Theorem 5.2** [COSI 89]. If \( p \) is an MSIP in \( N_p \), then for all markings \( M_0 \) of \( N \) there exists \( m_0(p) \) such that \( L(N_p, M_0 \cup m_0(p)) = L(N, M_0) \) (i.e. the addition of \( p \) with this number of tokens preserves the firing sequences).

That is, for any marking \( M_0 \), if \( p \) contains a certain number of tokens it does not constrain the language of the net. It is then an implicit restriction, a restriction that is implied by other places of the net.

Places \( p_3, p_4, p_5, p_6 \) in the Marked graph of figure 7 are all MSIPs. For instance, \( p_3 = p_4 + p_5 + p_1 \) (we identify here a place with its corresponding row in the incidence matrix). \( p_3 \) constrains the language with one token, but (as the reader can check) it does not constrain it with two.

The most important conceptual property of MSIPs for the purpose of this paper is that in Free Choice nets they are the **only way of adding and removing concurrency** in the form of a new place which can be later refined, while preserving SL&SB. But this is a property that we wish to preserve, since all LBFC nets are SL&SB by proposition 4.1.

This property of MSIPs requires a preliminary lemma. We profit from the definition of conservativeness, which was introduced for this purpose.

![Figure 7. Places p3, p4, p5, p6 are all marking structurally implicit.](image)

**Lemma 5.1.** Let \( N \) and \( N_p \) be two nets. Any two of the following three statements imply the third:

(a) \( N \) is conservative
(b) \( N_p \) is conservative
(c) \( p \) is an MSIP.

**Proof.** We only prove the part \((a \land b) \Rightarrow (c)\). The other two are proven by essentially the same argument. By definition, there exist \( Y_1 > 0 \) and \( Y_2 > 0 \) such that:

\[
Y_1^T.C = 0 \quad (1)
\]

\[
Y_2^T.C_p = 0 \quad (2)
\]
As $C_p = \left[ C \right]_p$, (2) can be rewritten as

\[(Y_2)^T C + k.l_p = 0\]  \hspace{1cm} (3)

where $k > 0$. Take $Y = \frac{1}{k}(\lambda Y_1 - Y_2)$, where $\lambda$ is large enough to make $Y > 0$. It is easy to see from (1) and (3) that $Y^T C = l_p$, as we wished to prove.

**Theorem 5.3.** Let $N$ and $N_p$ be two FC nets. Any two of the following three statements imply the third:

(a) $N$ is SL&SB
(b) $N_p$ is SL&SB
(c) $p$ is an MSIP.

**Proof.** $(a \land b) \Rightarrow (c)$. By proposition 2.1, both $N$ and $N_p$ are conservative. Apply then lemma 5.1.

$(a \land c) \Rightarrow (b)$. By definition of structural liveness, there exists $M_0$ such that $\langle N, M_0 \rangle$ is live. By definition of SIP, there exists $m_0(p)$ such that $\langle N_p, M_0 \cup m_0(p) \rangle$ has the same language than $\langle N, M_0 \rangle$, and therefore such that $\langle N_p, M_0 \cup m_0(p) \rangle$ is live. This proves that $N_p$ is structurally live. By proposition 2.1 and lemma 5.1, $N_p$ is conservative and thus structurally bounded.

$(b \land c) \Rightarrow (a)$. By definition of structural liveness, there exists $M_0 \cup m_0(p)$ such that $\langle N_p, M_0 \cup m_0(p) \rangle$ is live. Since $p$ is an MSIP, increasing the marking of $p$ if it is necessary, we can reach a marking $M_0 \cup m(p)$ of $N_p$ at which $p$ is implicit. By theorem 3.4, $\langle N_p, M_0 \cup m(p) \rangle$ is also live and bounded. Then, since the language is preserved by the removal of $p$, $\langle N, M_0 \rangle$ is live and $N$ is structurally live. By proposition 2.1 and lemma 5.1, $N_p$ is conservative and structurally bounded.

**Remark 5.2.** Notice that in the previous theorem the Free Choice property is required only for proving $(b \land c) \Rightarrow (a)$. This is the result that requires to use theorem 3.4.

Theorem 5.3 shows that in Free Choice nets places whose addition or removal preserve SL&SB must be **structurally redundant** (nonnegative linear combinations of other places). We can now present a pair of reduction/synthesis rules.

**Reduction Rule R2**

**Structural conditions:** $N$ is a Free Choice net containing an MSIP $p$

**Marking conditions:** every $P$-component of $N$ is marked at $M_0$.

**Structural changes:** $\bar{N}$ is obtained by removing $p$ from $N$ together with its input and output arcs.

**Marking changes:** $\forall p \in P: \bar{M}_0(p) = M_0(p)$.

**Proposition 5.1 R2 preserves liveness and boundedness.**

**Proof.** We have to show that the source marked net is live and bounded iff the target marked net is live and bounded.

$(\Rightarrow)$ By theorem 5.3, $\bar{N}$ is SL&SB. If we are able to show that every $P$-component of $\bar{N}$ is marked at $\bar{M}_0$, then we are done. But this is easy, because every $P$-component of $\bar{N}$ is a $P$-component of $N$ as well. Since $N$ is live and bounded by hypothesis, by theorem 3.1 all its $P$-components are marked and the result follows.

$(\Leftarrow)$ The source net can be obtained from the target net through the addition of an MSIP. Then, by theorem 5.3, the source net is SL&SB. Moreover, every $P$-component of the source net is
marked by the marking condition for the application of R2. By theorem 3.1, the source net is live and bounded.

Synthesis Rule S2

Structural conditions: \( N \) is Free Choice.
Marking conditions: none
Structural changes: \( \tilde{N} \) is obtained adding an MSIP \( p \) to \( N \) in such a way that \( \tilde{N} \) is Free Choice.
Marking changes: \( \forall \tilde{p} \in \tilde{P}, \tilde{p} \neq p: \tilde{M}_0(\tilde{p})=M_0(\tilde{p}) \). The marking \( \tilde{m}_0(p) \) of the new place \( p \) is chosen as follows: If there exists a \( P \)-component \( \tilde{N}_1 \) of \( \tilde{N} \) containing \( p \) such that \( M_0(\tilde{P}_1\setminus\{p\}) = 0 \), then choose \( \tilde{m}_0(p) > 0 \). Otherwise choose \( \tilde{m}_0(p) \) arbitrarily.

S2 preserves liveness and boundedness. The proof is similar to the one of proposition 5.1.

Notice that if the added MSIP is marked, the new net will always be live. But even if the MSIP contains no tokens, the new net can be live. Figure 7 is an example of the first case: \( p_3 \) cannot be added without tokens, since in this case the net is killed. An example of the second case is given in figure 8, where the addition of \( p_3 \) without tokens preserves liveness and boundedness.

Remark 5.3. Three limitations of R2 and S2 should be pointed out:
(1) R2/S2 are non-local rules. That is, global properties of the net must be checked in order to know if the rule is applicable.
(2) Both R2 and S2 do not preserve the bound of the net. The net of figure 9 is an example. Consider the net shown there, but without the place \( p \). The place \( p \) satisfies the conditions of application of S2. Nevertheless, if we add it the net is no longer safe.

![Diagram](https://via.placeholder.com/150)

Figure 8. The place \( p_3 \) can be added without tokens, and still preserve liveness and boundedness.

![Diagram](https://via.placeholder.com/150)

Figure 9. \( p \) is a 2-bounded MSIP, while the other places are safe.
(3) Both R2 and S2 do not preserve liveness and boundedness in general nets. The proof is left to the reader.

6. THE SYNTHESIS PROCEDURE
In this small section we introduce our synthesis procedure. As can be expected, we propose to start with a very simple Free Choice net which is trivially live and bounded, and then repeteadly apply to it the synthesis rules S1 and S2 described in the previous section.

Definition 6.1. A marked net $<N, M_0>$, where $N=(P, T; F)$ is atomic iff it is isomorphic to $([p], [t], ([p, t], (t, p)))$ and $M_0(p)>0$.

Following [GETH 84], we call the set of nets that can be synthetised from an atomic one by means of S1 and S2 well-formed. More formally:

Definition 6.2. The class of well formed Free Choice nets is the smallest class of marked nets given by:

1. Atomic marked nets are well formed
2. If $<N, M_0>$ is well formed and $<\tilde{N}, \tilde{M}_0>$ is a Free Choice net obtained by applying S1 or S2 to $<N, M_0>$, then $<\tilde{N}, \tilde{M}_0>$ is well formed.

Our task is to prove that the synthesis procedure is consistent and complete. The first part was in fact already done in the past section. The second, more complex, is considered in the next.

Theorem 6.1 (consistency of the synthesis procedure). Every well formed net is live and bounded.
Proof. Atomic nets are live and bounded and both S1 and S2 preserve liveness and boundedness.

Let us present now the synthesis of a net using our synthesis procedure. We have chosen a net with some history on it, namely the one used in [HACK 72] as main example. We have simplified it slightly to reduce the number of synthesis steps. The net is shown in figure 10.

![Figure 10. An LBFC net.](image-url)
Figure 11 shows some stages of the synthesis, which starts with an atomic net. The place of this atomic net is refined by $S_1$ (shaded places of the second net) and then $\text{MSIP}_1 = p_3 + p_6 + p_8$ is added. Thereafter $\text{MSIP}_1$ is refined and $\text{MSIP}_2 = p_1 + p_4 + p_9$ added (third net). Finally, $\text{MSIP}_2$ is refined and $\text{MSIP}_3 = p_{11} = p_7 + p_{10}$ added.

\[\begin{array}{c}
\text{Nodes of a subnet created from a place by } S_1 \\
\text{Marking structurally implicit places}
\end{array}\]

Figure 11. Synthesis of the net of figure 10.

7. COMPLETENESS OF THE SYNTHESIS PROCEDURE

The aim of this section is to show that the synthesis procedure of section 6 allows to synthetise all LBFC nets. Loosely speaking, we must show that the reduction rules R1 and R2 suffice to reduce every LBFC net to an atomic one.
We say a marked net $<N, M_0>$ is Ri-reducible iff Ri can be applied to $<N, M_0>$. $<N, M_0>$ is reducible iff there exists Ri such that $<N, M_0>$ is Ri-reducible. Our task is to show that every non-atomic LBFC net is reducible.

7.1 Reduction of maximal private connected subnets.
This subsection is concerned with the structure of MPC subnets in the P-components of LBFC nets. We show that these subnets have one single way-out place and therefore, by the Free Choice property, one single sink. It will be then easy to prove that these subnets can be reduced to a place by means of R1. At the end of the section the implications of these results are discussed.

Along the rest of the subsection $<N, M_0>$ denotes an LBFC net, $\mathcal{N}$ a minimal cover of P-components of $N$, $N_i=(P_i, T_i; F_i)$ an element of $\mathcal{N}$ and $N_i^*=(P_i^*, T_i^*; F_i^*)$ one of the MPC subnets of $N_i$. $\bar{N}_i$ denotes the environment of $N_i$. Given a place $p$ of $N_i^*$, the set of way-out places such that $p$ is connected to them by $F_i^*$-paths (which could be the empty path if $p$ is itself a way-out place) is denoted by WO(p).

We are interested in the P-components $N_i$ such that $\bar{N}_i$ is strongly connected. It is easy to see that there always exists at least one P-component satisfying this property.

Lemma 7.1 Let $N_i$ be such that $\bar{N}_i$ is strongly connected. Let $p$ be a way-in place of $N_i^*$. Then $\lvert\text{WO}(p)\rvert=1$.

Proof. Assume $\lvert\text{WO}(p)\rvert>1$. Notice that this implies $p\not\in\text{WO}(p)$, because if $p\in\text{WO}(p)$, then by the Free Choice property $\lvert p^* \rvert=1$ and $\text{WO}(p)\{p\}$, against the hypothesis.

Let $t$ be a source of $N_i$. By theorem 3.2, there exists a T-component of $N$ containing $t$. Consider two cases:

Case 1. Every T-component of $N$ containing $t$ contains also WO(p).

Let $p_1\in\text{WO}(p)$. By the Free Choice property, $p_1^*\{t_1\}$ and, since $t_1$ is a rendez-vous, $t_1\in T_i^*$. Therefore, no $F_i^*$-path leading from $p$ to an element of WO(p) different from $p_1$ can contain $p_1$. Taking into account that $N_i$ is strongly connected, a simple graph theoretical argument permits to prove the following: there exists a circuit $\Gamma$ of $N_i$ that contains $t$ and one and only one element of WO(p). Assume w.l.o.g. that this element is $p_1$. By theorem 3.3, there exists a T-component $N^1$ of $N$ containing $\Gamma$. Then $\Gamma\subseteq N_i\cap N^1$. As $N^1$ is strongly connected and $N^1$ contains WO(p) by hypothesis, there exists a TP-bridge $B$ from $\Gamma$ to $p_2\in\text{WO}(p)$, $p_2\neq p_1$. But as both $\Gamma$ and $p_2$ are contained in the same P-component $N_i$, there also exists a PP-bridge $B'$ from $p_2$ to $\Gamma$. Then $H=B;B'$ is a TP-handle of $\Gamma$ and, by theorem 3.1, $N$ is not live and bounded, against the hypothesis.

Case 2. There exists a T-component $N^1$ of $N$ that contains $t$ but does not contain WO(p).

It is easy to see that $N^1$ contains at least one place of WO(p). Let it be $p_1$, and let $p_2$ be one of the places of WO(p) that are not in $N^1$. Consider the net $\hat{N}=N^1\cap\bar{N}_i$ (recall that $\bar{N}_i$ is the environment of $N_i$).

By proposition 4.2, $\hat{N}$ is a set of pairwise disjoint T-components of $\bar{N}_i$, in this case nonempty since $\hat{N}$ contains at least transition $t$. Call $\bar{N}_i^1$ the T-component contained in $\hat{N}$ to which $t_1$ belongs. As $p_2\in\text{WO}(p)$, there exists a sink $t_2\in p_2^*$. A TT-bridge $B\subseteq N_i^1$ from $t_1$ to $t_2$ (and therefore from $\bar{N}_i^1$ to $t_2$) exists. On the other hand, as $\bar{N}_i$ is strongly connected, there exists a
bridge $B'$ contained in $\overline{N}_i$ from $t_2$ to $\overline{N}_i^1$. Since $\overline{N}_i^1$ is a T-component of $\overline{N}_i$, the last node of $B'$ must be a place. This implies that $H=B;;B'$ is a TP-handle of $\overline{N}_i^1$. We can then apply a proposition proved in [ESSI 89], saying that if a T-component has a TP-handle there is a circuit of it that also has a TP-handle. Then, by theorem 3.1, $N$ is not live and bounded, against the hypothesis.

**Theorem 7.1.** Let $N_i$ be such that $\overline{N}_i$ is strongly connected. Then $N_i'$ has exactly one way-out place.

**Proof.** We make the following claims:

1. $\forall p, p' \in P_i'$ such that there exists an $F_i'$-path from $p$ to $p'$: $WO(p') \subseteq WO(p)$

**Proof of claim 1.** Obvious from the definition of $WO(p)$.

2. $\forall p \in P_i'$: $|WO(p)| \leq 1$

**Proof of claim 2.** Every place of $N_i'$ is connected to at least one way-in place by an $(F_i')^{-1}$ path. Use then lemma 7.1 and claim 1.

3. $\forall p, p' \in P_i'$ such that there exists an $F_i'$-path from $p$ to $p'$: $WO(p') = WO(p)$

**Proof of claim 3.** Use claims (1) and (2).

4. $\forall p, p' \in P_i'$: $WO(p') = WO(p)$

**Proof of claim 4.** Since $N_i'$ is connected, there exists an $F_i \cup (F_i')^{-1}$ path from $p$ to $p'$. Apply repeatedly claim 3 to the $F_i'$ subpaths of it. The desired result follows immediately from claim 4.

**Theorem 7.2.** Let $N_i$ be such that $\overline{N}_i$ is strongly connected. Either $N_i'$ is isomorphic to $(\{p\}, \emptyset; \emptyset)$ or $N_i'$ is reducible to a place.

**Proof.** Assume $N_i'$ is not isomorphic to $(\{p\}, \emptyset; \emptyset)$. We show that $N_i'$ satisfies the three conditions of definition 5.1. The first two are obvious. The third one follows easily from theorem 7.1.

The net obtained from $<N, M_0>$ by reducing for every $P$-component $N_i \in \mathcal{R}$ the subnets of $MPC(N_i)$ (when they are not already composed by one single place) to the corresponding macroplaces is denoted by $<N_\mathcal{R}, M_\mathcal{R}>$. Since $R_1$ preserves liveness and boundedness, it follows that $<N_\mathcal{R}, M_\mathcal{R}>$ is an LBFC net. The theory developed in this section is not only helpful to prove the synthesis procedure complete. It also provides good insight into the behaviour of LBFC nets. MPC subnets of a sequential process represent that part of its behaviour that it can perform independently, without having to agree with other processes. We see now that this private behaviour is strongly constrained. Since by theorem 7.1 the MPC subnets have one single sink, the environment, once a token has been put into the subnet, knows that eventually this token will reach the way-out place of the subnet. The process can delay this final outcome, maybe for ever if the MPC subnet contains cycles and no fairness assumption is taken, but cannot choose between different outcomes. Freedom of choice requires to pay a high price: no process can take privately a decision which could have influence on the environment. This statement is rather reasonable: if nobody can be forced to do anything that she or he does not want to do, then nobody should be able to decide privately things that concern other people. Considered this way, this subsection only formalises this simple idea, giving a precise interpretation of the concepts concern and privacy.

If the reader wishes to keep in mind a graphical image of how MPC subnets in LBFC nets behave, we suggest to pay attention to the analogy between them and flipper machines (see figure 12).

A (slightly simplified) flipper machine has several way-in for the balls but one single way-out. The ball can spend a lot of time bouncing in rubber strips and bumpers but eventually (under a fairness assumption) will leave the board through the way-out. Moreover, no balls are created...
nor destroyed on the board, as happens in MPC subnets because they are P-graphs. We can therefore finish the section stating the following pseudotheorem: \textit{MPC subnets of LBFC nets are flipper subnets.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure12}
\caption{MPC subnets of LBFC nets are flipper subnets.}
\end{figure}

### 7.2 A characterisation of MSIPs in LBFC nets

After section 7.1, we must show that if \(<N_R, M_R>\) is live and bounded, it is either atomic or can be further reduced by means of R2. This can be achieved by showing that if it is non-atomic, then it contains an MSIP. This is possible because in LBFC nets MSIPs have a very simple and nice characterisation that is shown now.

We recall some of the notations introduced in section 5.2. \(N_p\) denotes the net obtained adding a place \(p\) to a net \(N\). If \(M\) is a marking of \(N\), \(M \cup m(p)\) denotes the marking of \(N_p\) whose restriction to \(N\) is \(M\) and puts \(m(p)\) tokens in \(p\).

**Theorem 7.3.** Let \(<N_p, M_0 \cup m_0(p)\>\) be an LBFC net. The place \(p\) is an MSIP iff \(N\) is strongly connected.

**Proof.** (\(\Rightarrow\)) \(<N, M_0>\) can be obtained from \(<N_p, M_0 \cup m_0(p)\>\) by means of R2. By proposition 5.1 \(<N, M_R>\) is an LBFC net. Using then a well known result, namely that all live and bounded nets are strongly connected (see [BEST 87]), \(N\) is strongly connected

(\(\Leftarrow\)) We claim that \(N\) satisfies conditions (a) and (b) of theorem 3.1. Before proving the claim, let us show that it leads to the desired result. If the claim is true, by theorem 3.1 \(N\) is lively and boundedly markable. By proposition 4.1, \(N\) is SL&SB as well. Since \(N_p\) is SL&SB as well by proposition 4.1, applying theorem 5.3 it follows that \(p\) is an MSIP.
Proof of the claim.
(i) No forking path of any strongly connected P-graph N' of N can be extended (in N) to a TP-handle.
This is obvious, because it cannot be extended to a TP-handle in N_p by theorem 3.1.
(ii) Every branching path of every strongly connected P-graph N' of N can be extended (in N) to a PP-handle.
In fact we are going to prove a statement from which (ii) can be derived: there is a P-component of N containing N'. It should be clear that, if this is true, in the process of getting the P-component from N, all branching paths of N' were extended to PP-handles.
Assume that N' cannot be extended to a P-component of N. Then there exists a maximal strongly connected P-graph N'=(P', T'; F') of N such that N'⊆N'' and N'' is not a P-component of N. As <N_p, M_o∪m_0(p)> is live and bounded, by theorem 3.3 there is a P-component of N_p containing N''. This means that there is a PP-handle H=(p_1,..., p_r) of N'' in N_p, and this handle must contain p, because otherwise N'' would not be maximal in N. Since N is strongly connected, p'=∪(t) by the Free Choice property and therefore H must contain t as well. This implies that t∈T'', because otherwise H would not be a handle of N''. Call B=(t,..., p_r) the subpath of H leading from t to p_r. Since N is strongly connected, there exists also a bridge B'⊆N from N'' to t. Then H'=(B'; B)⊆N is a handle of N'' ending at a place p_r. If H' starts at a transition, then H is a TP-handle of N'' and by theorem 3.1 N_p is not live and bounded, contradicting our hypothesis. If H' starts at a place, then N''∪H is a strongly connected P-graph of N, contradicting the maximality of N''. As we reach a contradiction in both cases, there is a P-component of N containing N'.

7.3 The reduction algorithm.
After the results contained in sections 7.1 and 7.2, we are ready to propose formally a reduction algorithm for LBFC nets, and prove that it can reduce any LBFC net to an atomic one.

Reduction algorithm

begin
input:=<N, M_o>, a marked net.
i:=0; <N_i, M_i><N, M_o>;
do while (<N_i, M_i> is reducible)
choose nondeterministically an applicable rule R_k
let <N_{i+1}, M_{i+1}> be the result of applying R_k to <N_i, M_i>;
i:=i+1
od;
<N, M_o><N_i, M_i>;
output <N, M_o>.
end.

Lemma 7.2 Let <N, M_o> be an LBFC net. <N, M_o> is either atomic or reducible.
Proof. By theorem 3.2, there exists a cover R of N, which can be chosen minimal. Consider the following two cases:
Case 1. |R|=1. In this case, N is a strongly connected P-graph. Then either <N, M_o> is atomic or R1-reducible (recall remark 5.1).

Case 2. |R|>1. It is easy to see that there exists N_i∈R such that N_i is strongly connected.

Clearly, N=N_i∪N_ij, where the union is on the elements of MPC(N_i). If <N, M_o> is non-R1-reducible, then N_ij=((p_j, ø; ø) for all N_ij∈MPC(N_i) (otherwise N_ij would be reducible to a
place by theorem 7.2). But then, as N is SL&SB and \( \tilde{N}_i \) is strongly connected, theorem 7.3 can be applied to show that all these places \( p_j \) are MSIPs. As moreover all P-components of N are marked at \( M_0 \), all these places satisfy the conditions of application of R2. Hence, \(<N, M_0>\) is R2-reducible.

**Theorem 7.4** (soundness and completeness of the synthesis procedure). Let \(<N, M_0>\) be a marked Free Choice net. \(<N, M_0>\) is live and bounded iff it is well formed.

**Proof.** \((\Rightarrow)\) Assume \(<N, M_0>\) is live and bounded. For both R1 and R2, the number of nodes of the target marked net is less than the number of nodes of the source marked net. This guarantees that the reduction algorithm applied to \(<N, M_0>\) terminates. By lemma 7.2, the output is an atomic marked net \(<\tilde{N}, \tilde{M}_0>\). This means that a sequence of application of R1 and R2 reduces completely \(<N, M_0>\). Consider the reverse sequence in which R1 and R2 are replaced by S1 and S2. It is clear that the application of this sequence to \(<\tilde{N}, \tilde{M}_0>\) yields \(<N, M_0>\).

\((\Leftarrow)\) See theorem 6.1.

8. **CONCLUSIONS**

A synthesis theory that produces all and only LBFC nets has been introduced. The theory can be called top-down, since starting from an atomic net a progressively more complex system is produced stepwisely. The two guidelines of our design were the search of simplicity and the use, while possible, of transformation techniques already known. This has required the connection of parts of net theory which have usually been kept rather disconnected. In particular, we refer to the analysis techniques based on graph theory, of which examples here are macroplace reductions or handles and bridges theory, and the ones based on linear algebra such as marking structurally implicit places.

We consider that the synthesis procedure proposed has three merits:

- it is simple, since it consists only of two rules. In fact, it is even minimal: both rules are necessary.
- it is efficient, because the conditions of application of the rules can be checked in polynomial time.
- it is meaningful and intuitive, because the two rules play very clear and in fact orthogonal roles. The macroplace rule refines a place into a structurally sequential nondeterministic subnet, but does not change the degree of concurrency of the structure. The marking structurally implicit place rule performs the other way round, taking charge of adding concurrency to the structure without changing its degree of nondeterminism or sequentiality.

It is important to point out that the two rules given here have two reverse-dual counterparts, which consist of:

- the synthesis of transitions into Marked Graphs
- the addition of new transitions, nonnegative linear combinations of transitions of the net.

They are not contained in this work, but the interested reader will find them in [ESPA 90]. It is immediate to show using Hack's duality theorem that they constitute another sound and complete set for the synthesis of LBFC nets. We have then tools for refining and adding both places and transitions, which is a very satisfactory result.

Our work was inspired and is in debt to the work of Genrich and Thiagarajan on Bipolar Schemes. Nevertheless, it is not easy to compare both. The reason is our different choices about the property to be preserved: safety in the case of [GETH 84] and boundedness in our case. Our work permits to synthesise a well-defined class of well behaved nets, broader than well behaved Bipolar Schemes, while needing a smaller set of rules. But these advantages derive partially from this choice, which allowed us to dissociate the search of good behaviour into two stages which correspond to structure and marking, and work on them independently. We could then introduce into the study of the structure stage very powerful tools, in particular those of algebraic nature.
Recent results by Desel [DESE 90] and Kovalyov [KOVA 91] complement and are complemented by our work. Although in both papers emphasis is put on reduction more than in synthesis, we interpret them here from a synthesis point of view, for the purpose of a comparison. In [DESE 90], the class of LSFC nets without frozen tokens is studied. The absence of frozen tokens can be seen as a particular kind of fairness. The importance of this class of nets lies, apart from the interest of this extra property, in the conjecture (still open) saying that this class is equivalent in a strong sense to the class of well behaved Bipolar Schemes. The author provides a set of four very simple local rules which synthesise all and only the nets in this class and which, in case the conjecture is true, would yield a simpler set of rules than the presented in [GETH 84]. There exists a trade-off between these results and ours: the rules are nicer, but the atomic systems are more complicated, namely the union of live and safe Marked Graphs and live and safe State Machines.

The class of so called α-nets is considered in [KOVA 91]. They are LBFC nets satisfying a structural condition. This restriction permits to obtain a sound and complete set of four local synthesis rules. A trade-off exists again: the rules in [KOVA 91] are local, but do not work for all LBFC nets. The author conjectures that in fact all LBFC nets can be synthesised using these rules. In our opinion, this is unlikely: The structural restriction defining α-nets makes them, loosely speaking, centralised systems, and this is probably essential for locality. The proof or disproof of this conjecture is a challenging open problem.

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