Probabilistic Cellular Automata

Carlos Camino
Outline of the Presentation

1. Cellular Automata
2. Majority Problem
3. Tool
4. Analysis
5. Results
6. Conclusions
1. Cellular Automata
1. Cellular Automata

Cells \((n = 10)\)

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
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</thead>
</table>
1. Cellular Automata

Neighborhood \((r = 1)\)
1. Cellular Automata

States \((s = 2)\)

= 0

= 1
1. Cellular Automata

Rule

[Diagram of cellular automata transitions]
1. Cellular Automata

Configuration

```
<table>
<thead>
<tr>
<th>t = 0</th>
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<tbody>
<tr>
<td>01</td>
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Rule

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1. Cellular Automata

Configuration

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Rule
2. Majority Problem
Example 1: \textit{solved}
Example 2: unsolved
### 2. Majority Problem

Some important solutions for $r = 3$ and $n = 149$:

<table>
<thead>
<tr>
<th>Year</th>
<th>Authors</th>
<th>Method</th>
<th>Performance</th>
</tr>
</thead>
<tbody>
<tr>
<td>1978</td>
<td>Gács, Kurdyumov, Levin</td>
<td>human-written</td>
<td>81.6%</td>
</tr>
<tr>
<td>1994</td>
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</tr>
</tbody>
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Question: Does a perfect rule exist?
2. Majority Problem

Variation 1: (Capcarre, Sipper and Tomassini - 1996)

Change the output specification:

If the initial configuration contains more 1’s (or 0’s) than 0’s (or 1’s), no two cells with state 0 (or 1) can coexist in the final configuration.
2. Majority Problem

Variation 1: (Capcarre, Sipper and Tomassini - 1996)

2. Majority Problem

Variation 2: (Fuks - 1997)

Use two Cellular Automata:

Combine the use of Rule 184 and Rule 232.
First apply only Rule 184, then only Rule 232.
Variation 2: (Fuks - 1997)

Use two Cellular Automata:

Combine the use of Rule 184 and Rule 232.
First apply only Rule 184, then only Rule 232.
2. Majority Problem

Variation 2: (Fuks - 1997)

Rule 184

Rule 232
3. Tool
3. Tool

```
Problem type
Density threshold
Final configuration
Maximum time units
Lattice size
Initial density
Fraction of randomly distributed cells
Initial configuration
Number of rules
Success probability
Uniform rule application
Probabilities
Rules

Experiment

Result

Time units
```
3. Tool

- Problem type
- Density threshold
- Final configuration
- Maximum time units
- Lattice size
- Initial density
- Fraction of randomly distributed cells
- Initial configuration
- Number of rules
- Success probability
- Uniform rule application
- Probabilities
- Rules
- Number of experiments

Statistic
- Number of correctly solved runs
- Number of incorrectly solved runs
- Number of not terminated runs
- Average running time
3. Tool

```
Problem type
Density threshold
Final configuration
Maximum time units
Lattice size
Initial density
Fraction of randomly distributed cells
Initial configuration
Number of rules
Success probability
Uniform rule application
Probabilities
Rules
Number of experiments
Number of statistics
Variable

Graph

Number of correctly solved runs in each statistic
Number of incorrectly solved runs in each statistic
Number of not terminated runs in each statistic
Average running time in each statistic
```
4. Analysis
4. Analysis

![Statistic Visualization](image1)

- **Goodness**
  - Binomial Distribution Probability

![Statistic Report](image2)

- **STATISTIC**
  - Number of Tests: 80
  - Variable: Binomial Distribution Probability

- **TEST**
  - Number of Experiments: 5000

- **PROBLEM**
  - Problem Type: Majority Problem
  - Density Threshold: 0.5
  - Minimum runtime: 

- **CONFIGURATION**
  - Size: 
  - Density: 
  - Degree of randomness: 
  - Cells: random

- **RULE**
  - Number of rule instances: 2
  - Binomial distribution probability: variable
  - Apply rule instance uniformly: yes
  - Rule instances probabilities: to be set
  - Rule instances number: [184, 232]
  - Lookup table:
    - 11: 1, 10: 1, 100: 1, 101: 0, 010: 0, 011: 0
    - 111: 0, 110: 1, 101: 1, 100: 0, 010: 0, 011: 0
    - 1: 1, 4: 4, 0: 1, 0: 0

![Average Running Time](image3)

- **Average Running Time**
  - Binomial Distribution Probability

![Graph](image4)
Total running time as the sum of the running time of two phases:
4. Analysis

Phase 1 simulated with \( n = 30 \), \( d = 2/3 \) and \( p = 0.4 \):

- **Green**: Rule 184
- **Red**: Rule 232
4. Analysis

Phase 1 modelled as a random-walker with stochastic matrix:

\[ P = \begin{pmatrix}
p & q & 0 & 0 & 0 & \ldots & 0 & 0 \\
p & 0 & q & 0 & 0 & \ldots & 0 & 0 \\
0 & p & 0 & q & 0 & \ldots & 0 & 0 \\
0 & 0 & p & 0 & q & \ldots & 0 & 0 \\
p & q & 0 & p & 0 & \ldots & 0 & \vdots \\
0 & 0 & 0 & \ldots & 0 & \ddots & q & 0 \\
0 & 0 & 0 & \ldots & 0 & p & 0 & q \\
0 & 0 & 0 & \ldots & 0 & 0 & p & q \\
\end{pmatrix} \]

(q = 1–p)
4. Analysis

Expected number of steps from state 0 to state \( z-1 \) for some \( z \):

\[
h_{0,z-1} = \begin{cases} 
0, & \text{if } z = 1, \\
1q^{-1}, & \text{if } z = 2, \\
1q^{-1} + 1q^{-2}, & \text{if } z = 3, \\
2q^{-1} + 0q^{-2} + 1q^{-3}, & \text{if } z = 4, \\
2q^{-1} + 2q^{-2} - 1q^{-3} + 1q^{-4}, & \text{if } z = 5, \\
3q^{-1} + 0q^{-2} + 3q^{-3} - 2q^{-4} + 1q^{-5}, & \text{if } z = 6, \\
3q^{-1} + 3q^{-2} - 3q^{-3} + 5q^{-4} - 3q^{-5} + 1q^{-6}, & \text{if } z = 7, \\
4q^{-1} + 0q^{-2} + 6q^{-3} - 8q^{-4} + 8q^{-5} - 4q^{-6} + 1q^{-7}, & \text{if } z = 8, \\
\vdots & \text{for higher } z.
\end{cases}
\]
4. Analysis

Expected number of steps from state 0 to state $z-1$ for some $z$:

$$h_{0,z-1} = \begin{cases} 
0, & \text{if } z = 1, \\
1q^{-1}, & \text{if } z = 2, \\
1q^{-1} + 1q^{-2}, & \text{if } z = 3, \\
2q^{-1} + 0q^{-2} + 1q^{-3}, & \text{if } z = 4, \\
2q^{-1} + 2q^{-2} - q^{-3} + 1q^{-4}, & \text{if } z = 5, \\
3q^{-1} + 0q^{-2} + 3q^{-3} - 2q^{-4} + 1q^{-5}, & \text{if } z = 6, \\
3q^{-1} + 3q^{-2} - 3q^{-3} + 5q^{-4} - 3q^{-5} + 1q^{-6}, & \text{if } z = 7, \\
4q^{-1} + 0q^{-2} + 6q^{-3} - 8q^{-4} + 8q^{-5} - 4q^{-6} + 1q^{-7}, & \text{if } z = 8, \\
\vdots & 
\end{cases}$$

And as a recursion:

$$h_{0,0} = 0,$$
$$h_{0,z} = (h_{0,z-1} + z)q^{-1} - h_{0,z-1}.$$
Phase 2 simulated with $n = 30$, $d = 2/3$ and $p = 0.3$:

- **Green**: Rule 184
- **Red**: Rule 232
5. Results
5. Results

Running Time against $p$ with $d = 0.3$ and $n = 10$
5. Results

Running Time against $d$ with $p = 0.4$ and $n = 100
5. Results

Running Time against \( n \) with \( d = 0.1 \) and \( p = 0.4 \)
6. Conclusions