PSPACE Complete Problems

**QSAT**

QSAT = Set of Quantified CNF Expressions which evaluate to true

\[ \exists \forall x, \exists y, \exists z, \ldots, \exists v, Q_{x,y,z,\ldots,v}(x_1, x_2, \ldots, x_n) \Rightarrow true? \]

Is it true, that we can chose a truth-value for \( x_i \) such that for all truth-values for \( x_j \), we can choose a ...... such that \( \phi(x_1, x_2, \ldots, x_n) \) evaluates to true?

**QSAT is PSPACE-complete**

**Membership (I)**

\[ \exists x, \forall y, \exists z, \ldots, \exists v, Q_{x,y,z,\ldots,v}(x_1, x_2, \ldots, x_n) \Rightarrow true? \]

The idea is to implicitly traverse the tree of all possible truth assignments to \( x_1, x_2, \ldots, x_n \).

However, this tree has linear depth and exponential size. Therefore, we can only store our position in the tree -- similar to the idea of Savich’s Theorem.

**Membership (II)**

\[ \exists x, \forall y, \exists z, \ldots, \exists v, Q_{x,y,z,\ldots,v}(x_1, x_2, \ldots, x_n) \Rightarrow true? \]

We use the Reachability Method:

\[ \langle G^x_{s,t}, s, t \rangle \text{ is a REACH instance with } C^x_{\max}(t) \text{ nodes.} \]

\[ \langle G^{u,v}_{s,t}, s, t \rangle \text{ is a REACH instance with } C^{u,v}_{\max}(t) \text{ nodes.} \]

**Hardness (I)**

\[ \langle G^x_{s,t}, s, t \rangle \text{ is a REACH instance with } C^x_{\max}(t) \text{ nodes.} \]

\[ \langle G^{u,v}_{s,t}, s, t \rangle \text{ is a REACH instance with } C^{u,v}_{\max}(t) \text{ nodes.} \]

\[ f(\alpha) = n^t \text{ give us:} \]

\[ \langle G^x_{s,t}, s, t \rangle \text{ is a REACH instance with } C^x_{\max}(t) \text{ nodes.} \]

\[ \langle G^{u,v}_{s,t}, s, t \rangle \text{ is a REACH instance with } C^{u,v}_{\max}(t) \text{ nodes.} \]

\[ f(\alpha) = n^t \text{ give us:} \]

\[ \langle G^x_{s,t}, s, t \rangle \text{ is a REACH instance with } C^x_{\max}(t) \text{ nodes.} \]

\[ \langle G^{u,v}_{s,t}, s, t \rangle \text{ is a REACH instance with } C^{u,v}_{\max}(t) \text{ nodes.} \]

**Hardness (II)**

\[ \langle G^x_{s,t}, s, t \rangle \text{ is a REACH instance with } C^x_{\max}(t) \text{ nodes.} \]

\[ \langle G^{u,v}_{s,t}, s, t \rangle \text{ is a REACH instance with } C^{u,v}_{\max}(t) \text{ nodes.} \]

\[ PATH[0](U, V) \Rightarrow (U = V) \text{ or } (U, V \in G^x_{s,t}) \]

\[ PATH[0](U, V) \Rightarrow \exists Z \in PATH[0](U, Z) \times PATH[0](Z, V) \]

\[ PATH[0](S, T) \Rightarrow G^x_{s,t} \in \text{ REACH} \]
QSAT is PSPACE-complete
Hardness (III)

\( \text{PATH}[0](U, V) \equiv (U = V) \lor (U, V \in G^n) \)

We can express \( \text{PATH}[0] \) as circuit of polynomial size (we will get rid of the circuit later)

\( \text{PATH}[i + 1](U, V) \equiv \exists Z(\text{PATH}[i](U, Z) \land \text{PATH}[i](Z, V)) \)

\( \text{PATH}[i + 1](U, V) \) would become exponentially large -- we have to reuse \( \text{PATH}[i] \).

QSAT is PSPACE-complete
Hardness (IV)

\( \text{PATH}[i + 1](U, V) \equiv \exists Z \forall Y[((X = U \land Y = Z) \lor (X = Z \land Y = V)) \Rightarrow \text{PATH}[i](X, Y)] \)

We can bring the quantifiers to the front (used only locally)

\( \text{PATH}[n^*](U, V) \equiv QX_1 QX_2 \ldots QX_n \forall C(U, V, X_1, X_2, \ldots, X_n) \)

We have to represent the above expression as a polynomially sized quantified CNF-expression.

\( \text{PATH}[n^*](U, V) \equiv QX_1 QX_2 \ldots QX_n \exists H \phi(U, V, X_1, X_2, \ldots, X_n, H) \)

QSAT is PSPACE-complete
Hardness (V)

\( \text{PATH}[n^*](U, V) \) is expressible as polynomially sized quantified CNF-expression in prenex form.

Thus, QSAT is PSPACE-hard, using \( \text{PATH}[n^*](S, T) \) as result of the reduction.

Geography

\[ \forall x \exists y \exists z [(x \lor y) \land (y \lor z) \land (\neg x \lor z)] \]

Geography = Set of \( < G, s > \) with \( G = \langle V, E \rangle \) directed and \( s \in V \)
where player A has a winning strategy;
\( \text{i.e.}, \) can force player B into a dead end.
### Proving Hardness & Completeness

#### Summary

<table>
<thead>
<tr>
<th>Problem</th>
<th>Complexity</th>
<th>Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>REACH</td>
<td>NL-complete</td>
<td>Reachability Method</td>
</tr>
<tr>
<td>CIREVAL</td>
<td>P-complete</td>
<td>Time-Table Method</td>
</tr>
<tr>
<td>CIRSAT</td>
<td>NP-complete</td>
<td>Time-Table Method</td>
</tr>
<tr>
<td>QSAT</td>
<td>PSPACE-complete</td>
<td>Reachability Method</td>
</tr>
</tbody>
</table>

*Inherently Sequential*

*Guess and Check – Optimization*

*Games – Optimal Strategies*