Turing Machines
A Model of Computation

Alan Turing
- Born in 1912
- 1922: Troubles in School
- 1936: Turing Machine
- WWII: Bletchley Park
- 1954: Suicide?!

Original Motivation
Modeling “Human Computers”
What are “Human Computers”? Remember the Turing Test!

Today’s Perspective
Modeling “Computers”
What are “Computers”?• Storage Device
  - read/write access
  - finite size (conceptually arbitrarily large)
• Control Unit
  - defines which step to do next
  - aka CPUs/Programs
Today's Perspective

Modeling “Computers”
What are “Computers”?

Modeling…
But what for???

Today's Perspective

Modeling “Computers”
What are “Computers”?

Reasoning about
- algorithms
- computational problems

Treating them as Mathematical Objects!!!

Today's Perspective

Modeling “Computers”
What are “Computers”?

Algorithms as Mathematical Objects

- TMs are meant for formulating & proving general statements about algorithms:
  - What is computable by a TM?
  - How much time/space does a TM need to solve a problem?
- TMs are NOT meant for:
  - programming
  - real computations
  - browsing the web

So what?!

Turing Machines and real computers?!

Relevance of TMs:
What is Computable by TMs?

LCMs can do anything that could be described as “rule of thumb” or “purely mechanical”.

This is sufficiently well established that it is now agreed amongst logicians that "calculable by means of an LCM" is the correct accurate rendering of such phrases.

(Turing in 1948 on his Logical Computing Machine)

Relevance of TMs:
Church-Turing Thesis

LCMs can do anything that could be described as “rule of thumb” or “purely mechanical”.

- Historically: A lot of models are equivalent to TMs (i.e., they describe the same set of algorithms)
  - Lambda Calculus
  - Partially Recursive Functions
  - ...
- Practically: All known computer systems are equivalent to TMs.
Relevance of TMs: How efficient are TMs?

All reasonable models of computation are polynomially related to the TM wrt. their time performance.

This is established by simulation arguments…

A Pentium 4 can be simulated by a TM with runtime \( n^k \) for some fixed \( k \).

Relevance of TMs: Extended Church-Turing Thesis

All reasonable models of computation are polynomially related to the TM wrt. their time performance.

This is established by simulation arguments…

…. it's a thesis – not a theorem.

DNA-Computing

Quantum-Computing

This Lecture

- Definition of TMs
- Execution of TMs
- Multi-Tape TMs
- Non-Deterministic TMs

Storage Device of a TM

- Tape
  - arbitrarily long but finite strip divided into cells
  - each cell contains a single symbol
  - finite alphabet of symbols

# # # 1 1 0 0 1 # # # # # # # # #

- Tape Head
  - accesses one cell at a time (active cell)
  - reads symbol from active cell
  - overwrites symbol to active cell
  - moves left, right or stays
Control Unit of a TM

- **Set of States**
  - finite size
- **Transition Function**
  given a state and an input symbol, the control decides which
  - symbol to write
  - direction the head is moved
  - new state to assume

### Example

Alphabet: \{0,1\}
Blank Symbol: #

```
1:1/ ←
0:0/ ←
1:0/ ←
0:1/ ←
#:1/ ←
#:0/ ←
#:#/ ←
```

Input String

```
#:#/
#:#/
#:#/
```

```
S
T
H
B
```

```text
##11001####
```

Example

```
0:0/
#:#/
```

```
1:1/
```

```
0:1/
```

```
0:0/
```

```
1:0/
```

```
1:1/
```

```
#:#/
```

Example

```
0:1/
```

```
0:0/
```

```
1:1/
```

```
#:#/
```

Example

```
0:0/
```

```
1:1/
```

```
#:#/
```

Example

```
0:1/
```

```
0:0/
```

```
1:1/
```

```
#:#/
```

Example

```
0:0/
```

```
1:1/
```

```
0:0/
```

```
#:#/
```

Example

```
0:1/
```

```
0:0/
```

```
1:1/
```

```
#:#/
```
Example

Example

Example

Example

Example

Formal Turing Machine Definition

\[ M = (K, \Sigma, \delta, s) \]

- finite set of states \( K \)
- finite set of symbols \( \Sigma \) (alphabet)
- transition function \( \delta : K \times \Sigma \rightarrow (K \cup \{H, A, R\}) \times \Sigma \times \{\leftarrow, \rightarrow, \downarrow\} \)
- initial state \( s \in K \)
Example

Alphabet: \{0,1\}
Blank Symbol: #

\[ M = \langle K, \Sigma, \delta, s \rangle \]

\[ \Sigma = \{0,1,\#\} \]
\[ K = \{S, T, B\} \]
\[ s = S \]

State H is never left!

Example

\[ \Sigma = \{0,1,\#\} \]
\[ K = \{S, T, B\} \]
\[ s = S \]

# is always included

Example

\[ \Sigma = \{0,1,\#\} \]
\[ K = \{S, T, B\} \]

This Lecture

• Definition of TMs
• Execution of TMs
• Multi-Tape TMs
• Non-Deterministic TMs
• Uniformity Theorem

Formal Turing Machine Definition

\[ M = \langle K, \Sigma, \delta, s \rangle \]

- \( K \) is a set of states.
- \( \Sigma \) is a finite set of symbols (including blank symbol).
- Transition function \( \delta \) accepts inputs \( (s, x) \) where \( s \in K \) and \( x \in \Sigma \).
- Initial state \( s \), initial tape contents, and initial position indicated.

Examples:

1. \( \Sigma = \{0,1,\#\} \)
   \[ K = \{S, T, B\} \]
   \[ s = S \]

2. \( \Sigma = \{0,1,\#\} \)
   \[ K = \{S, T, B\} \]
   \[ s = S \]

3. \( \Sigma = \{0,1,\#\} \)
   \[ K = \{S, T, B\} \]
   \[ s = S \]

4. \( \Sigma = \{0,1,\#\} \)
   \[ K = \{S, T, B\} \]
   \[ s = S \]
This Lecture

- Definition of TMs
- Execution of TMs
- Multi-Tape TMs
- Non-Deterministic TMs
- Uniformity Theorem

Execution of TMs

The execution of a TM is described formally as a Sequence of Configurations.

A Step of TM is the transition from one Configuration to the next one.

Two special configurations:
- Initial Configuration
- Halting Configuration

Configuration

A Configuration $C$ of $M = \langle K, \Sigma, \delta, s \rangle$ describes the entire state of $M$ during some execution.

- It must contain: cursor position, tape contents, state

<table>
<thead>
<tr>
<th>#</th>
<th>#</th>
<th>#</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>#</th>
<th>#</th>
<th>#</th>
<th>#</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>B</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>#</td>
<td>#</td>
<td>#</td>
<td>#</td>
</tr>
</tbody>
</table>

Triples $C = \langle q, w, u \rangle$ with $q \in K$ and $w, u \in \Sigma^*$
- $w$ is the string up until the tape head
- $u$ contains the rest
- $\#$s which have not been visited are ignored

A Computational Step

$C = \langle q, w, u \rangle$ and $\delta(q, w_i) = \langle q', w_i', \text{dir} \rangle$.

If $\text{dir} = \rightarrow$ then $w' = w_1...w_{i-1}w_i'w_{i+1}$

$u' = u_{i+1}...u_n$ (\#$s$ are padded)
A Computational Step

\[ C : q, w, u > \text{ and } \delta(q, w) = q', w', \text{dir} >. \]

If \( \text{dir} \rightarrow \) then \( w' = w_1 \cdots w_n \), \( u' = u_2 \cdots u_n \).

For \( \text{dir} = \) and \( \text{dir} = - \) analogously.

\( C \) is said to yield \( C' : q', w', u' >. \)

The transition from \( C \) to \( C' \) is a single step.
In our example, we started with the configuration $\langle s_1, 1001 \rangle$ and input $x \in (\Sigma - \#)^*$.

Given $M = \langle K, \Sigma, \delta, s \rangle$ and input $x \in (\Sigma - \#)^*$, $C = \langle s, x_1, x_2, \ldots, x_n \rangle$ is the initial configuration.

If $q = H$ then the M computed a function. The result is the string between the head and the first # to the right.

In our example: $M(x) = x + 1$
**Excursus: Decision Problems**

- Remember the UNO-Problem.
- Given a set of states, you can ask whether there is peaceful seating arrangement.
- This a decision problem: The answer is a single bit.
- Their simple structure is helpful within complexity.
- Formal Language: The set of positive instances.
- Functional problems can be "reduced" to decision problems.

**Halting Configuration: Decision**

If \( q = A \) (\( q = R \)) then \( M \) accepted (rejected) the input \( x \).

Such an \( M = \langle K, \Sigma, \delta, s \rangle \) decides a language:
\[
L(M) = \{ x \in \Sigma^* : M(x) = A \}
\]

**Runtime of a TM**

Let \( M = \langle K, \Sigma, \delta, s \rangle \) and \( x \in (\Sigma - \{\#\})^* \).

The number of steps between initial and halting configuration is the runtime of \( M \) on \( x \).

If \( M \) halts within \( f(|x|) \) steps or less for all \( x \in (\Sigma - \{\#\})^* \) then \( M \) runs in time \( f(n) \).

If \( M \) does not reach a halting state (H,A,R), then \( M \) does not terminate (runs forever).

**Example**

Runtime of \( M \) on 110110: 12 steps

**Space used by a TM**

Let \( M = \langle K, \Sigma, \delta, s \rangle \) and \( x \in (\Sigma - \{\#\})^* \).

The number of symbols in the largest configuration is the space required by \( M \) on input \( x \).

If \( M \) runs within \( f(|x|) \) space or less for all \( x \in (\Sigma - \{\#\})^* \) then \( M \) runs in space \( f(n) \).
Example

<table>
<thead>
<tr>
<th>#</th>
<th>#</th>
<th>#</th>
<th>#</th>
<th>1</th>
<th>1</th>
<th>0</th>
<th>1</th>
<th>0</th>
<th>#</th>
<th>#</th>
<th>#</th>
<th>#</th>
<th>#</th>
</tr>
</thead>
</table>

- \( C = < S, 1, 1001 > \)
- \( C = < S, 1, 1001 > \)
- \( C = < S, 11001 > \)
- \( C = < S, 11001 >, \varepsilon > \)
- \( C = < T, 11001, > > \)
- \( C = < T, 11000, > > \)
- \( C = < B, 11010, > > \)
- \( C = < B, 1, 11010, > > \)
- \( C = < B, 1, 11010, > > \)
- \( C = < H, \#, 11010, > > \)

Maximal configurations

Space requirement of \( M \) on input 110110:

7 cells

M runs in space \( n+2 \)

This Lecture

- Definition of TMs
- Execution of TMs
- Multi-Tape TMs
- Non-Deterministic TMs
- Encoding
- Constants do not matter

Multi-Tape TMs

- Instead of a single tape, we use several tapes
- They are dedicated:
  - Input Tape (read only)
  - Work Tapes (read/write)
  - Output Tape (write only)

Multi-Tape TMs: Definition

adapting the definition \( M = \{ K, \Sigma, \delta, s \} \):

If \( M \) is a \( k \) tape TM, then

\[ \delta : K \times \Sigma^* \rightarrow (K \cup \{H, A, R\}) \times \Sigma^* \times \{ <-, \rightarrow, \rightarrow \} \]

Read and write \( k \) symbols, move on \( k \) tapes

rest is the same.
Multi-Tape TMs: Configuration

Adapting $C = < q, w, u >$ with $q \in K$ and $w, u \in \Sigma^*$:

If $M$ is a $k$ tape TM, then

$C = < q_0, w_1, u_1, \ldots, w_k, u_k >$ with $q \in K$ and $w_i, u_i \in \Sigma^*$

... just $k$ tapes

Multi-Tape TMs: Space Bound

$C = < q, w_1, u_1, \ldots, w_k, u_k >$ with $q \in K$ and $w_i, u_i \in \Sigma^*$

The number of symbols in the largest configuration is the space required by $M$ on input $x$.

But only the contents of the work tapes are counted!

I.e., input and output are not considered for space bounds.

Multi-Tape TMs: Stronger??

Let $M$ be a $k$ - tape TM running in time $O(f(n))$. Then there is a 1- tape TM $M'$ with $M(x) = M'(x)$ which runs in time $O(f^2(n))$.

(On the other hand: Palindroms can be decided by a
• 2-tape TM within time $O(n)$
• 1-tape TM requires $O(n^2)$.)

This Lecture

• Definition of TMs
• Execution of TMs
• Multi-Tape TMs

• Non-Deterministic TMs
• Uniformity Theorem

Deterministic TMs

The TMs we saw so far were deterministic.

I.e., the input determined the outcome of the computation.

I.e., we used a transition function:

$\delta : K \times \Sigma \rightarrow (K \cup \{H, A, R\}) \times \Sigma \times \{\leftarrow, \rightarrow, \_\}$

That’s the way, our real computers work....
Non-Deterministic TMs

Non-Deterministic TMs are a formalism to express certain algorithms.

… but you **cannot** simulate a nondet. TM directly by a real computer…

We start with an example…

**Example: UNO**

Given an undirected graph $G = \langle V, E \rangle$.

Is there a circle which includes all nodes in $V$?

Sure:

![Graph](image)

Note: We are only looking at the decision problem …

**Example: UNO**

Given an undirected graph $G = \langle V, E \rangle$.

Is there a circle which includes all nodes in $V$?

If you try to solve this problem, you will end up enumerating the possible solutions…

1. for each permutation $\pi$ of $V$
2. if $\pi$ is a path in $G$ then accept
3. reject

Almost equivalently, you can write:

1. guess a permutation $\pi$ of $V$
2. if $\pi$ is a path in $G$ then accept
3. reject

… it might make a wrong guess, but if there exists a solution, at least one guess will find it!

A way to capture such “algorithms”: Non-deterministic TMs.
**Deterministic TMs**

We emphasized the fact that det. TMs use a transition function.

\[
δ : K \times Σ \rightarrow (K \cup \{H, A, R\}) \times Σ \times \{\leftarrow, \rightarrow, \cdot\}
\]

**Non-Deterministic and Deterministic TMs**

We emphasized the fact that det. TMs use a transition function.

\[
δ : K \times Σ \rightarrow (K \cup \{H, A, R\}) \times Σ \times \{\leftarrow, \rightarrow, \cdot\}
\]

For a reason. Non-det. TMs use a transition relation.

\[
δ \subseteq K \times Σ \times (K \cup \{H, A, R\}) \times Σ \times \{\leftarrow, \rightarrow, \cdot\}
\]

**Non-Deterministic TMs**

For a reason. Non-det. TMs use a transition relation.

\[
δ \subseteq K \times Σ \times (K \cup \{H, A, R\}) \times Σ \times \{\leftarrow, \rightarrow, \cdot\}
\]

Configurations are still the same:

\[
C = q, w, u \quad \text{with} \quad q \in K \quad \text{and} \quad w, u \in Σ^*
\]

But how does it run???

**A Deterministic Computational Step**

\[
C = q, w, u \quad \text{and} \quad δ(q, w) = q', w', \text{dir} > .
\]

If dir = \( \rightarrow \) then \( w' = w_1 \ldots w_n \quad \text{and} \quad u' = u_1 \ldots u_k \)

For dir = \( \rightarrow \) and dir = \( \rightarrow \) analogously.

\( C \) is said to yield \( C' = q', w', u' \).

The transition from \( C \) to \( C' \) is a single step.

**A Nondeterministic Computational Step**

\[
C = q, w, u \quad \text{and} \quad δ(q, w) = q', w', \text{dir} > .
\]

If dir = \( \rightarrow \) then \( w' = w_1 \ldots w_n \quad \text{and} \quad u' = u_1 \ldots u_k \)

For dir = \( \rightarrow \) and dir = \( \rightarrow \) analogously.

\( C \) is said to yield \( C' = q', w', u' \).

The transition from \( C \) to \( C' \) is a single step.

**Example**

\( C = S, \#, . \)

\[
\begin{array}{c}
S \quad \text{#} \quad \text{#} \quad \text{H}
\end{array}
\]

#1/→
#0/→
##/→

...
Example

\[ C = <S,\#,\epsilon > \]
\[ C = <H,\# ,\epsilon > \]
\[ C = <S,1\# ,\epsilon > \]
\[ C = <S,0\# ,\epsilon > \]

This Lecture

- Definition of TMs
- Execution of TMs
- Multi-Tape TMs
- Non-Deterministic TMs
- Uniformity Theorem

Uniformity Theorem

For every polynomial time algorithm \( A \), there is a family of circuits \( C_1, C_2, \ldots \), such that

- \( C_i \) can be constructed in time polynomial in \( i \)
- \( C_i(x) = A(x) \)

Example

\[ \text{##11001####} \]
\[ \text{##11001####} \]

Example

\[ \text{##11001####} \]
\[ \text{##11001####} \]
Example

Uniformity Theorem

Uniformity Theorem

Uniformity Theorem

Uniformity Theorem

Uniformity Theorem

Uniformity Theorem

Checks for accepting config → true/false

f(n) cells

f(n) configs
Uniformity Theorem for Nondeterministic Algorithms

For every nondeterministic polynomial time algorithm $A$, there is a family of circuits $C_1$, $C_2$, ..., such that:

- $C_i$ can be constructed in time polynomial in $i$.
- Note: Inputs of $C_i$ are of size polynomial in $i$.
- There exists a $y$ with $C_i(x,y) = \text{true}$ iff there is a computation of $A(x) = \text{true}$.

Example

$C = \langle S, \#, \epsilon \rangle$

$C = \langle H, \#, \epsilon \rangle$  $C = \langle S, 1\#, \epsilon \rangle$  $C = \langle S, 0\#, \epsilon \rangle$

Example

$C = \langle H, \#, \epsilon \rangle$

$C = \langle S, 1\#, \epsilon \rangle$  $C = \langle S, 0\#, \epsilon \rangle$

Example

$C = \langle H, \#, \epsilon \rangle$

$C = \langle S, 1\#, \epsilon \rangle$  $C = \langle S, 0\#, \epsilon \rangle$

Example

$C = \langle H, \#, \epsilon \rangle$

$C = \langle S, 1\#, \epsilon \rangle$  $C = \langle S, 0\#, \epsilon \rangle$
Uniformity Theorem

This Lecture

- Definition of TMs
- Execution of TMs
- Multi-Tape TMs
- Non-Deterministic TMs
- Uniformity Theorem