Black Ninjas in the Dark: Formal Analysis of Population Protocols

Javier Esparza

Joint work with Michael Blondin, Pierre Ganty, Stefan Jaax, Antonín Kučera, Jérôme Leroux, Rupak Majumdar, Philipp J. Meyer, and Chana Weil-Kennedy

Technical University of Munich

Deaf Black Ninjas in the Dark

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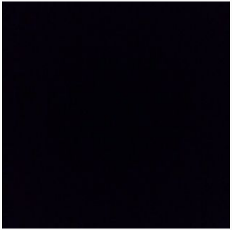

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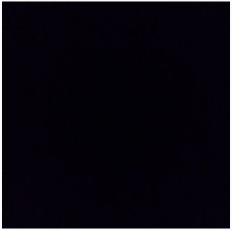
- Deaf Black Ninjas meet at a Zen garden in the dark
- They must decide by majority to attack or not (no attack if tie)
- How can they conduct the vote?



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- Additionally, they are active or passive .

attack
active

don't attack
active

attack
passive

don't attack
passive

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- Initially: all ninjas active, estimation = own vote.

Deaf Black Ninjas in the Dark

Goal of voting protocol:

- eventually all ninjas reach the same estimation, and
- this estimation corresponds to the majority.

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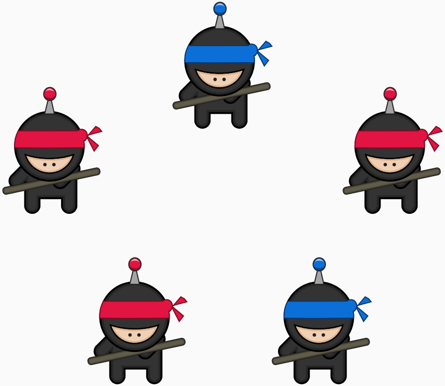
Graphically:

- Initially more red ninjas $\Longrightarrow$ eventually all ninjas red.
- Initially more blue ninjas or tie $\Longrightarrow$ eventually all ninjas blue.

Majority protocol: Are there more red ninjas than blue ninjas?



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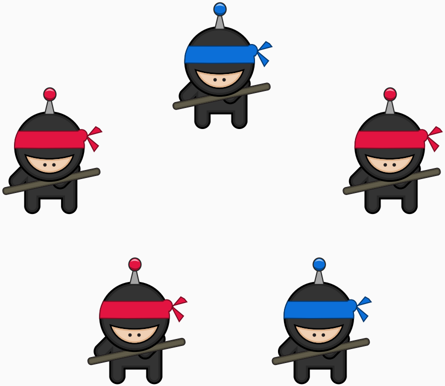
- Active ninjas of opposite colors become passive and blue



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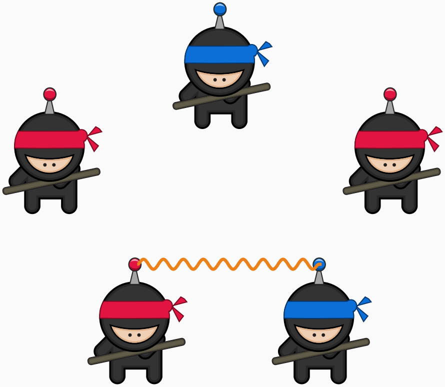

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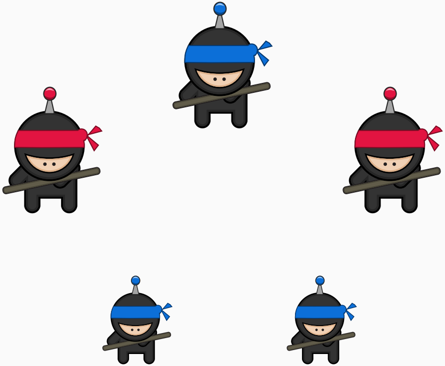

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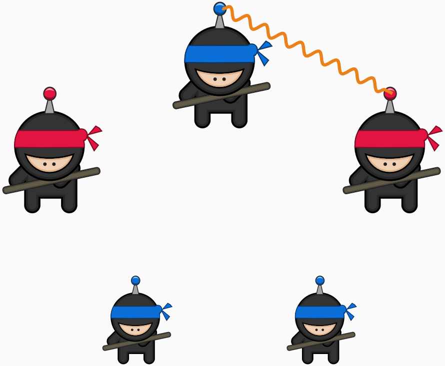

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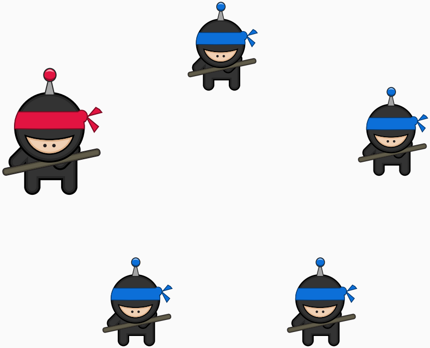

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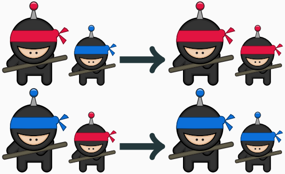

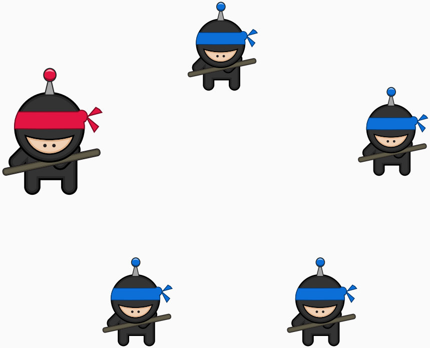

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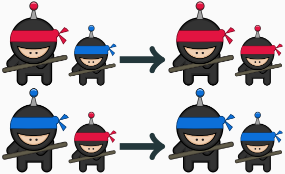

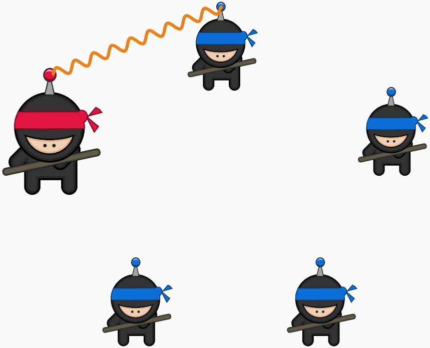

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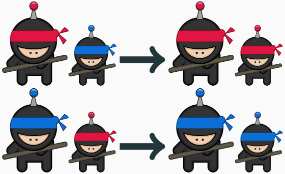

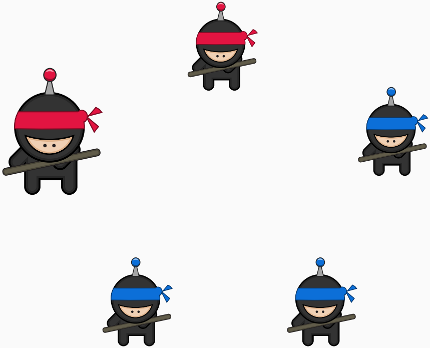

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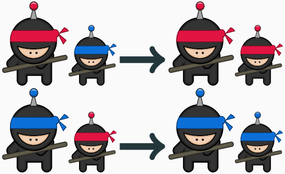

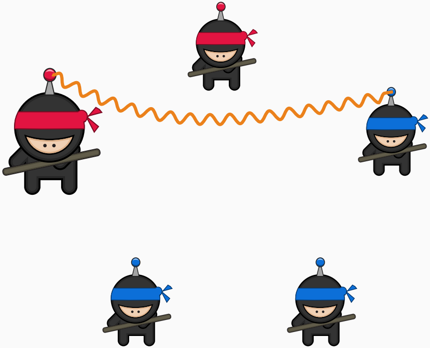

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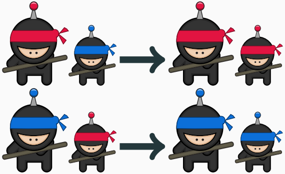

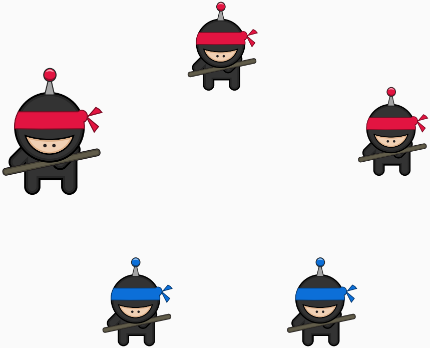

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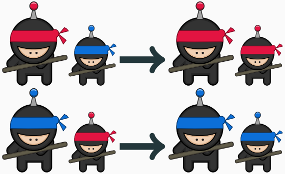

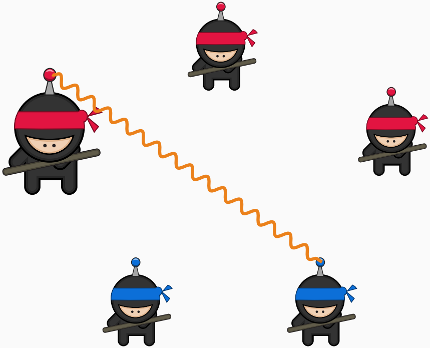

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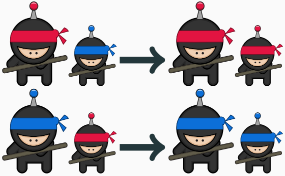

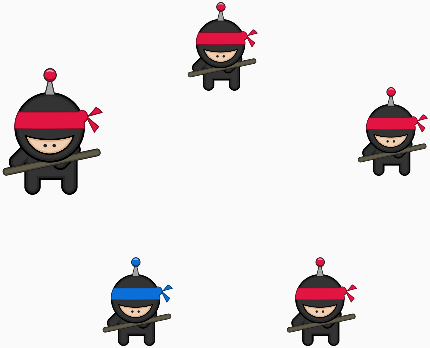

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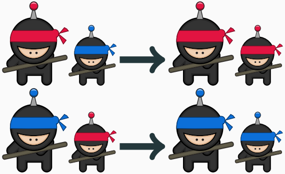

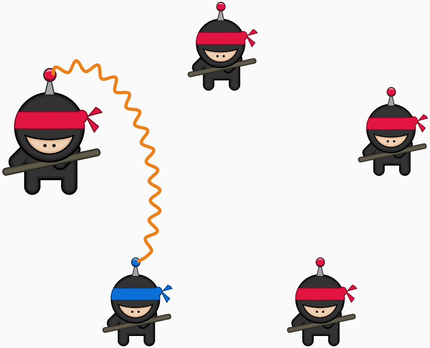

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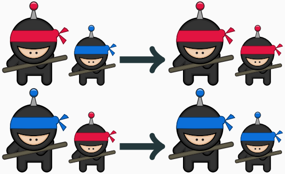

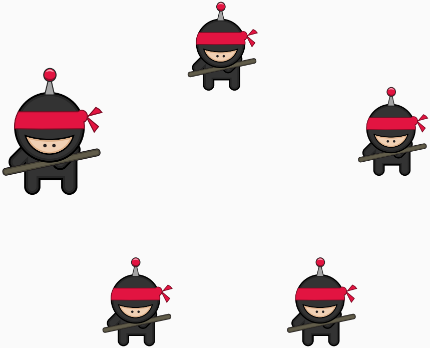

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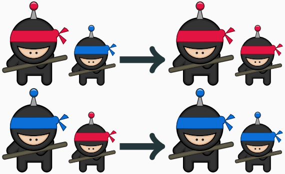
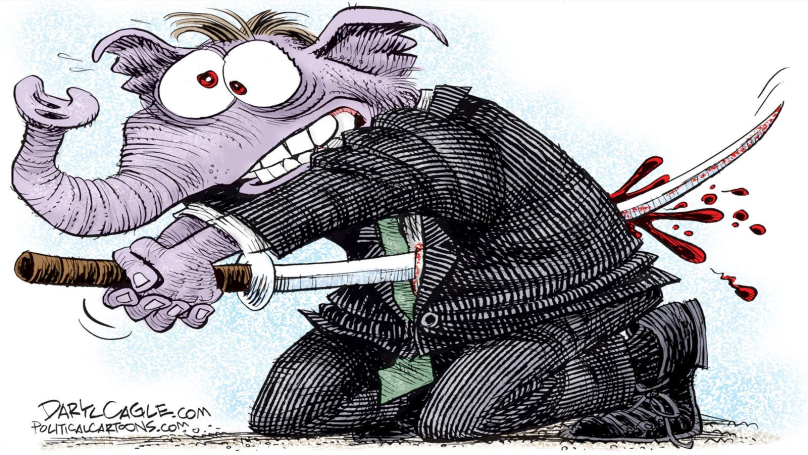


Sad story ...



Sensei II

Majority protocol: Why?

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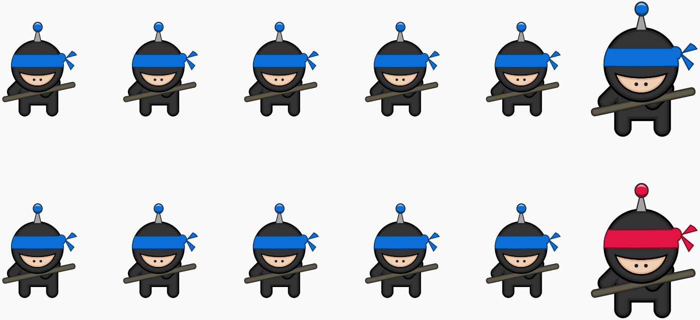

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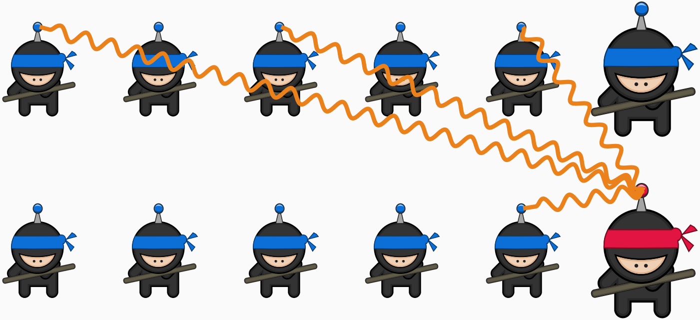

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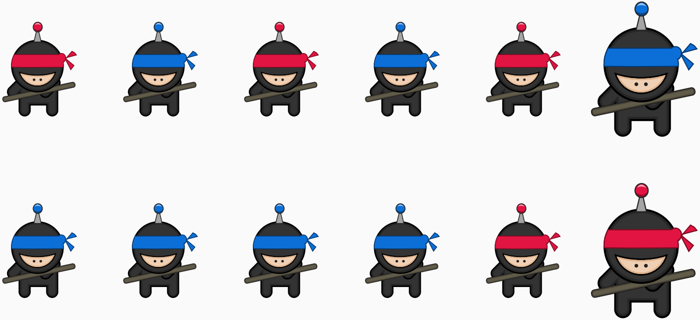

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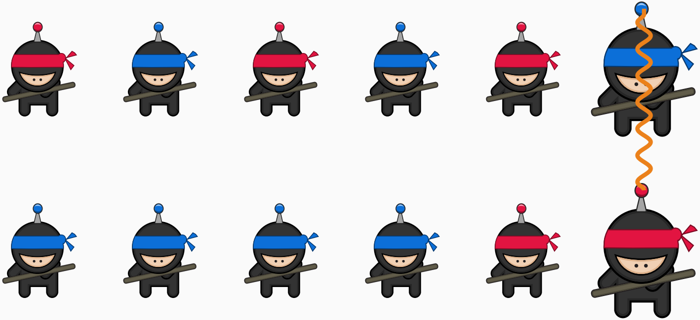

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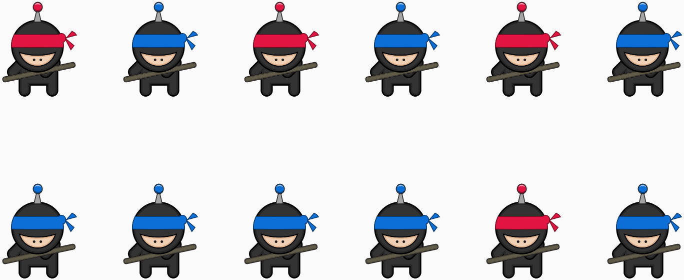

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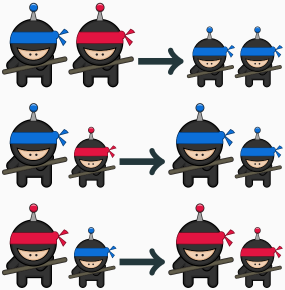
NO CONSENSUS!

Sensei II's protocol: Are there more red ninjas than blue ninjas?

Interaction rules:



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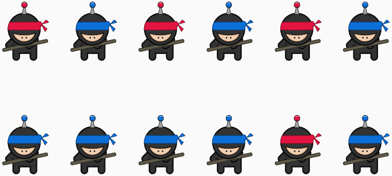

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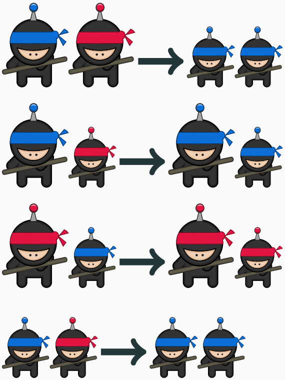

Passive blue ninjas convert passive red ninjas to their color

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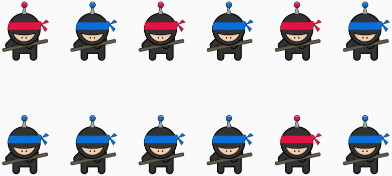

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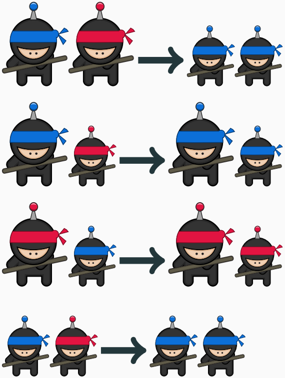

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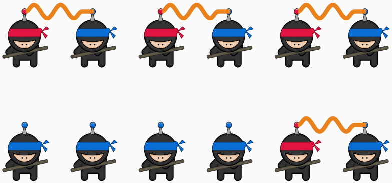

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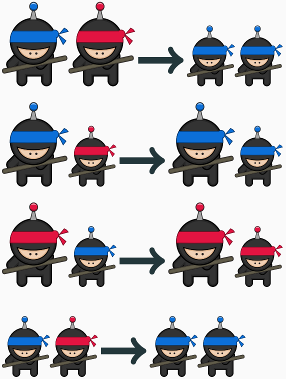

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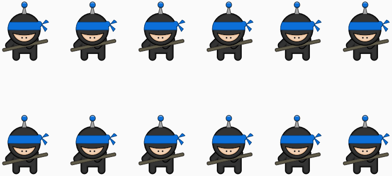

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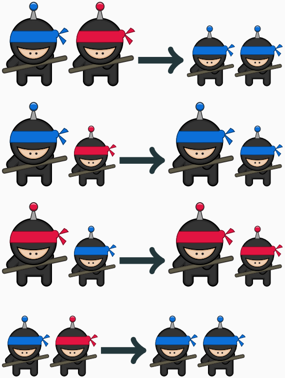

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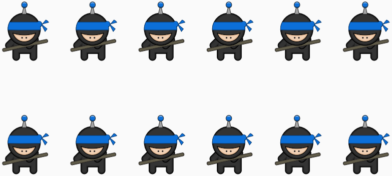
Go!



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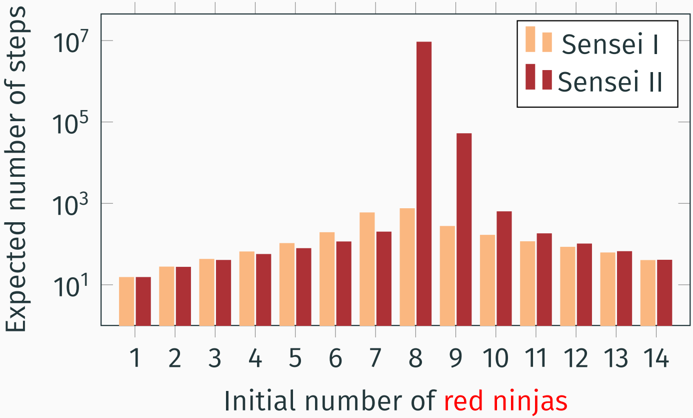


Expected number of steps to stable consensus for a population of 15 ninjas.

Very sad story ...

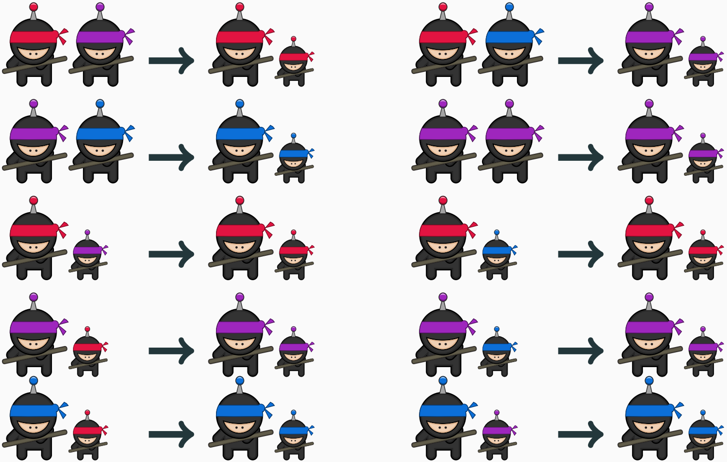
Sensei III

Sensei III’s protocol

 = Attack majority  = Don’t attack majority  = Tie

Interaction rules:

Go!



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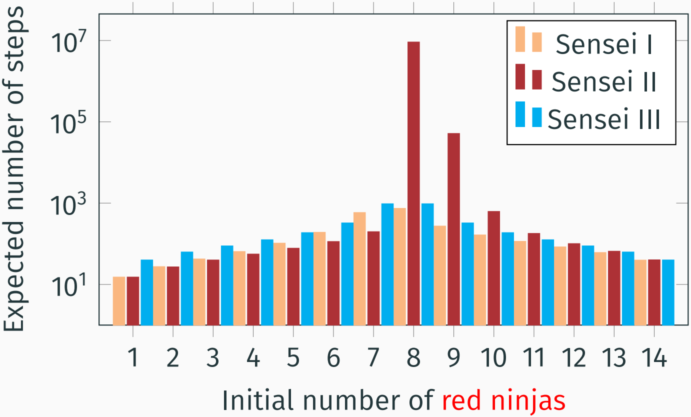

Expected number of steps to stable consensus for a population of 15 ninjas.

Sensei III's questions

Formalization questions:

- What is a protocol?
- When is a protocol "correct"?
- When is a protocol "efficient"?

Sensei III's questions

Verification questions:

- How do I check that my protocol is correct ?
- How do I check that my protocol is efficient ?

Sensei III's questions

Expressivity questions:

- Are there protocols for other problems?
- How large is the smallest protocol for a problem?
- And the smallest efficient protocol?

Formal model of distributed computation by collections of

identical, finite-state, and mobile agents

like

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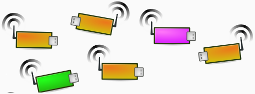

ad-hoc networks of mobile sensors

Population protocols

Angluin, Aspnes *et al.* PODC'04

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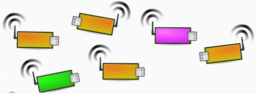

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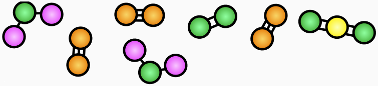

"soups" of molecules (Chemical Reaction Networks)

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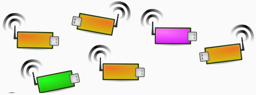

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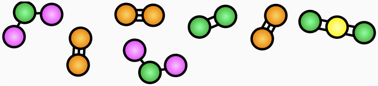

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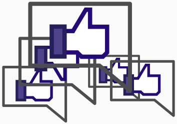

people in social networks

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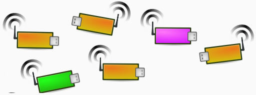

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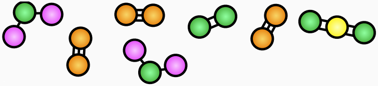

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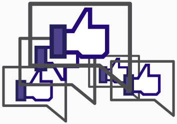

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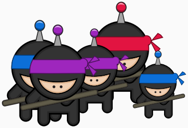

...and ninjas!

Population protocols: formal model

Angluin, Aspnes *et al.* **PODC'04**

- *States:* finite set Q
- *Opinions:* $O : Q \to \{0, 1\}$
- *Initial states:* $I \subseteq Q$
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Population protocols: formal model

Angluin, Aspnes *et al.* PODC'04

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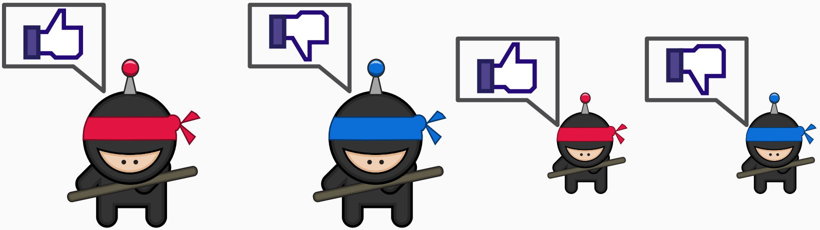

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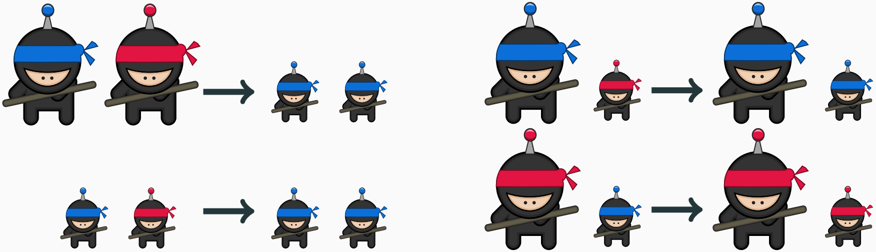

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$$\begin{pmatrix} 2 & 5 & 1 & 3 \end{pmatrix}$$

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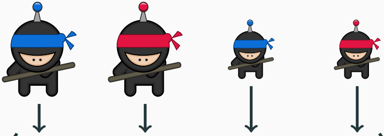
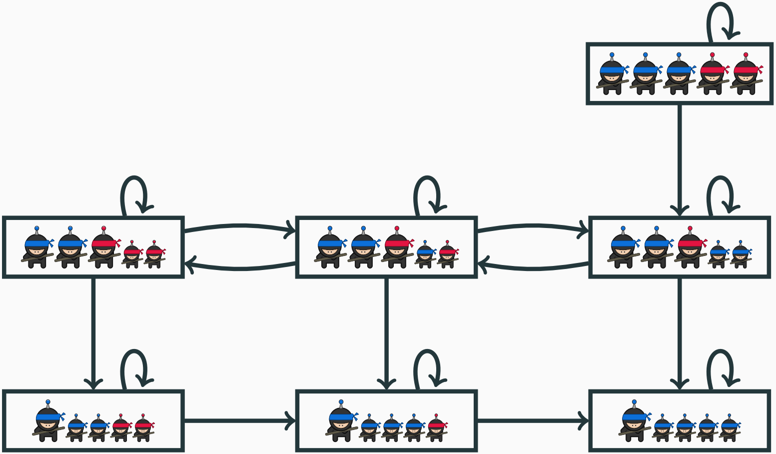


$$(2 \quad 5 \quad 0 \quad 0)$$

Population protocols: runs

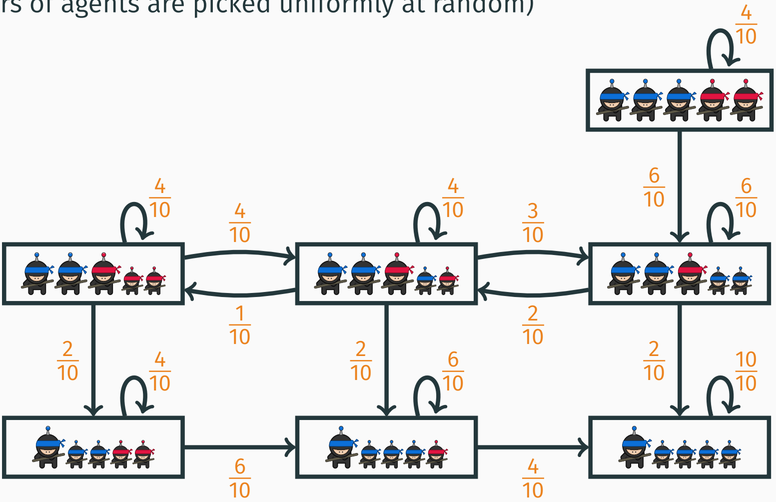
Reachability graph for $(3, 2, 0, 0)$**:**



Population protocols: runs

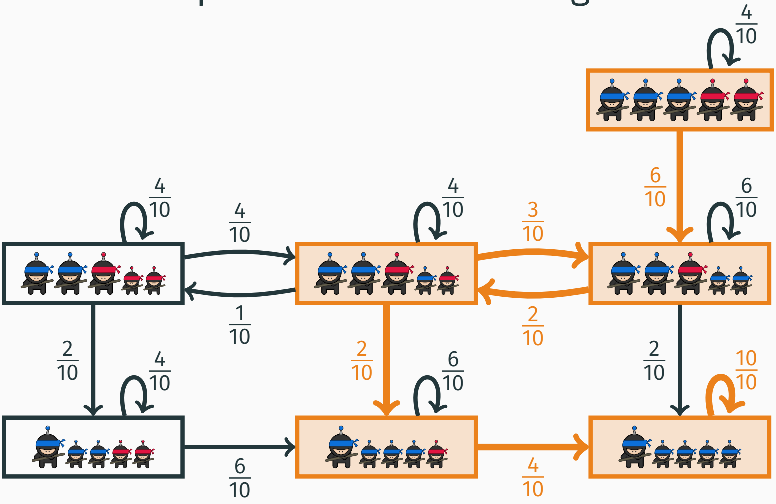
Underlying Markov chain:

(pairs of agents are picked uniformly at random)



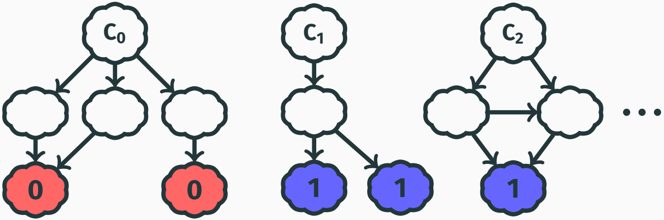
Population protocols: runs

Run : infinite path from initial configuration



Population protocols: computing predicates

Protocol computes φ: InitC $\rightarrow \{0, 1\}$:
for every $C \in \text{InitC}$, the runs starting at C
reach **stable consensus $\varphi(C)$** with probability 1.



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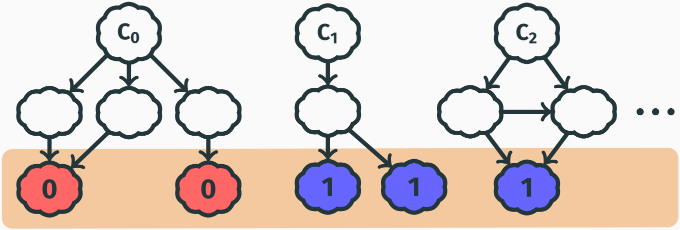

Protocol computes $\varphi(C_0) = 0, \varphi(C_1) = 1, \varphi(C_2) = 1, \ldots$

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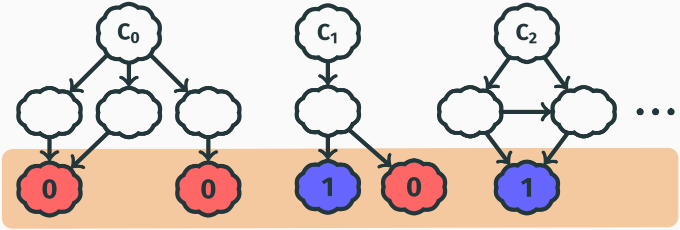

Protocol ill defined for C_1

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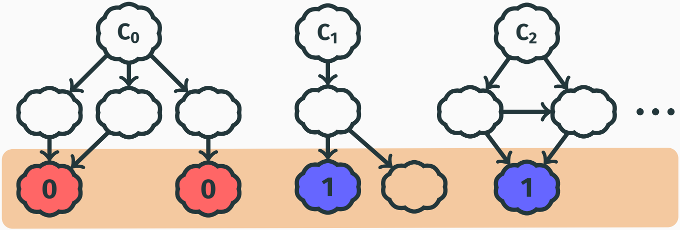

Protocol ill defined for C_1 (Sensei I's problem)

Population protocols: computing predicates

A protocol is well specified if it computes some predicate

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A protocol for a predicate φ is correct if it computes φ (in particular, correct protocols are well specified)

Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude ...

Expressive power

Angluin, Aspnes, Eisenstat Dist. Comp.'07

Population protocols compute precisely the predicates definable in Presburger arithmetic, *i.e.* $\mathrm{FO}(\mathbb{N}, +, <)$

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Since Presburger arithmetic has quantifier elimination, it suffices to:

- Exhibit PPs for threshold and modulo predicates

$$a_1 x_1 + \cdots + a_n c_n \leq b \qquad a_1 x_1 + \cdots + a_n c_n \equiv b \pmod{c}$$

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- Prove that computable predicates are closed under negation and conjunction

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Proof: PPs only compute Presburger predicates

- Much harder!

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- "Constructive" proof by E., Ganty, Leroux, Majumdar Acta Inf.'17

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Other variants considered:

- Approximate protocols — *e.g.* Angluin, Aspnes, Eisenstat DISC'07
- Protocols with leaders — Angluin, Aspnes, Eisenstat Dist. Comput.'08
- Protocols with failures — Delporte-Gallet *et al.* DCOSS'06
- Trustful protocols — Bournez, Lefevre, Rabie DISC'13
- Mediated protocols, etc. — Michail, Chatzigiannakis, Spirakis TCS'11

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Efficiency measured by the expected number of interactions until stable consensus: $\mathit{Inter}(n)$

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In a natural model: expected (parallel) time to consensus satisfies

$$Time(n) = Inter(n)/n$$

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Every Presburger predicate is computable by a protocol with $\mathit{Inter}(n) \in \mathcal{O}(n^2 \log n)$

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Every Presburger predicate is computable by a protocol with a leader and $\mathit{Inter}(n) \in \mathcal{O}(n \log^{O(1)}(n))$

Open whether $\mathcal{O}(n \log^{O(1)}(n))$ achievable without leaders.

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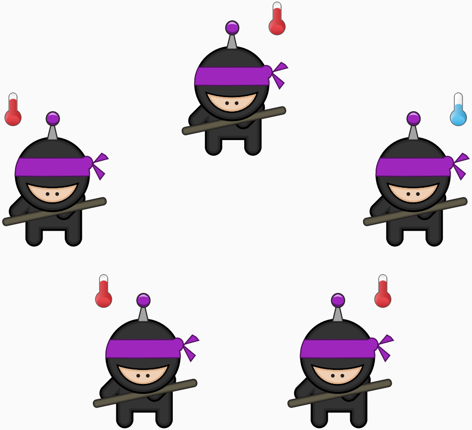
Succinctness–An Example

Protocol for: Are there at least 4 sick ninjas?



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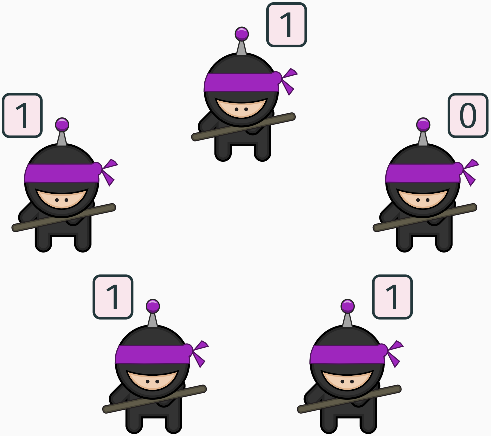

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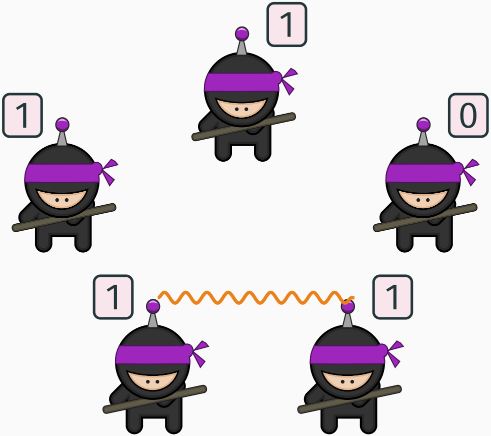

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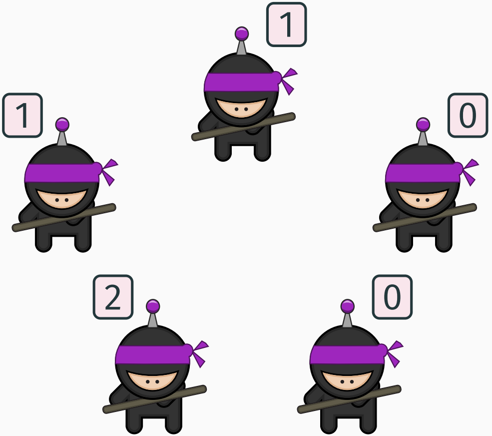

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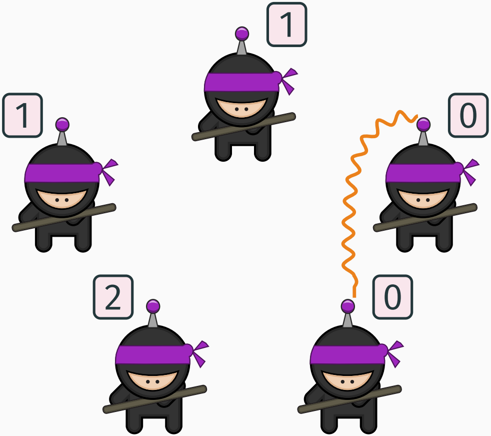

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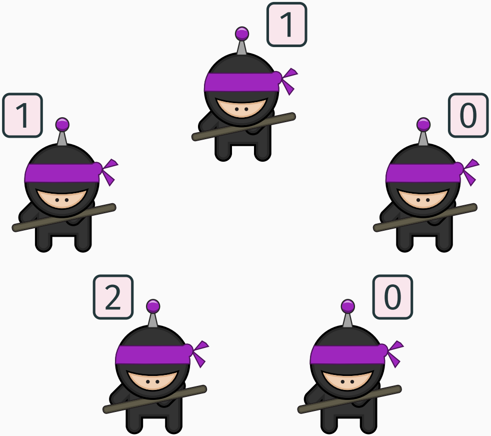

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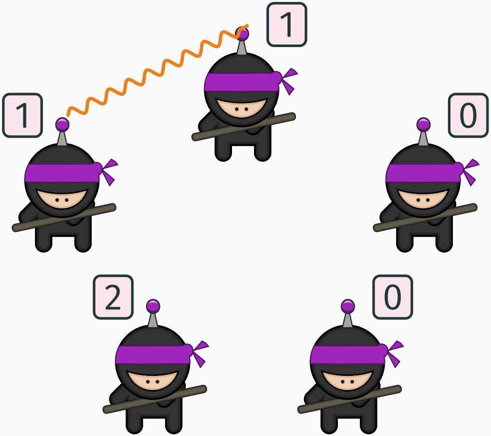

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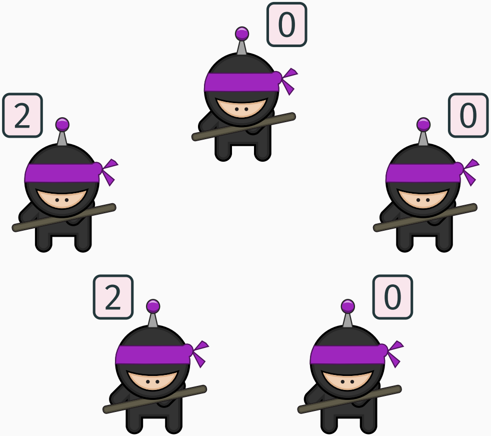

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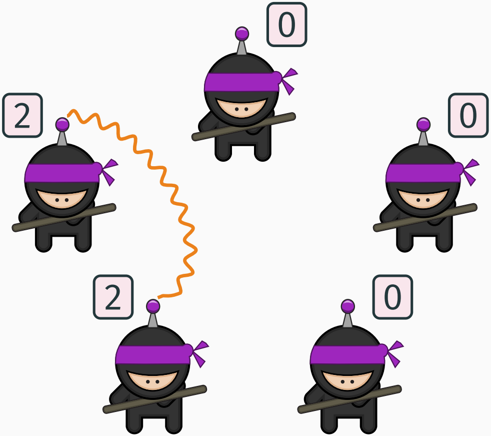

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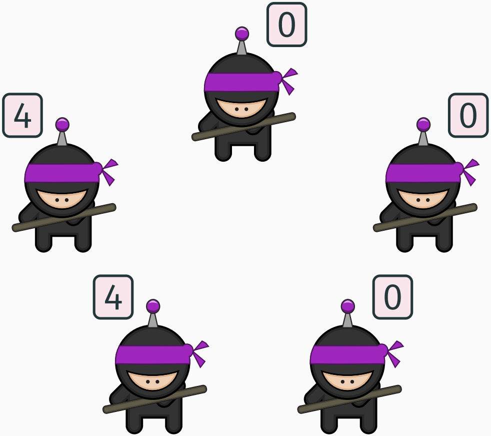

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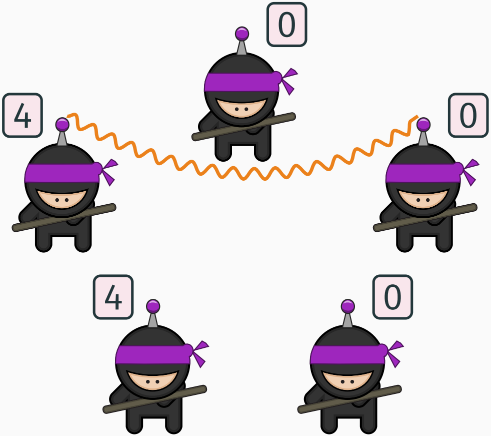

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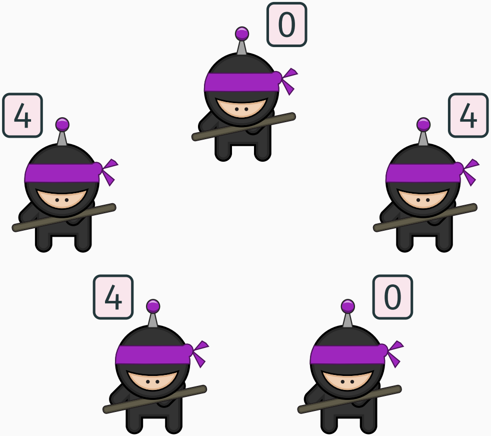

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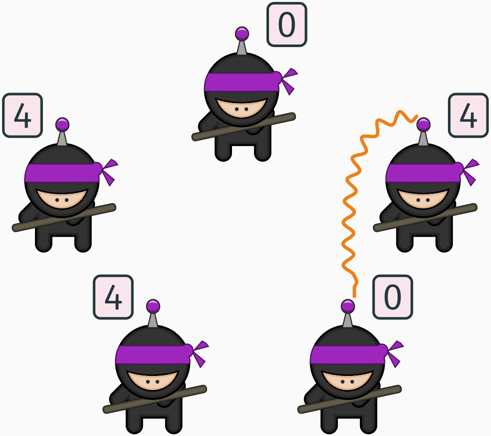

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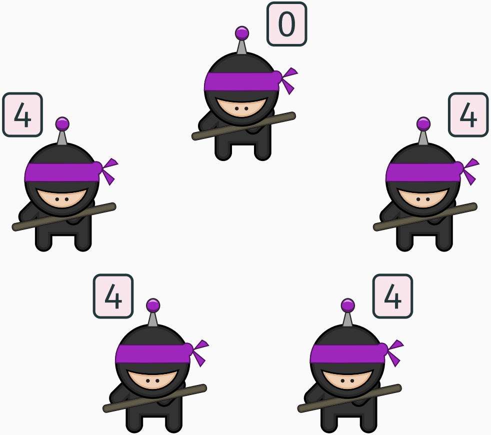

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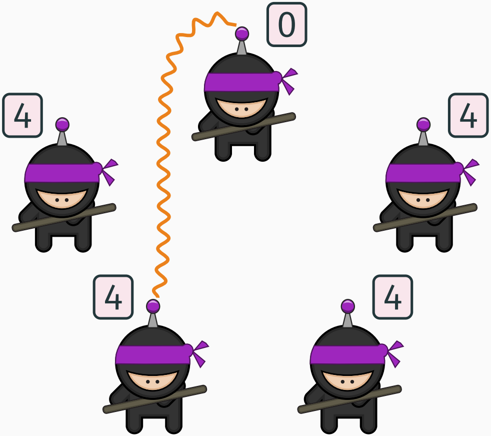

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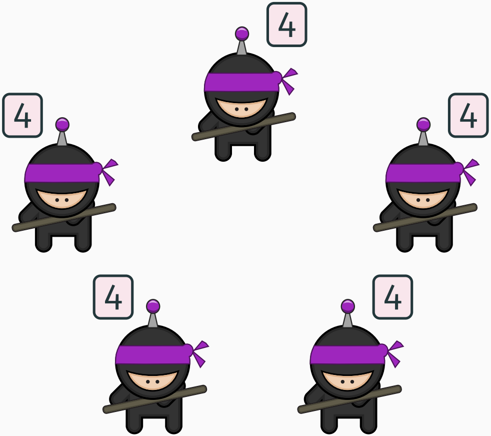

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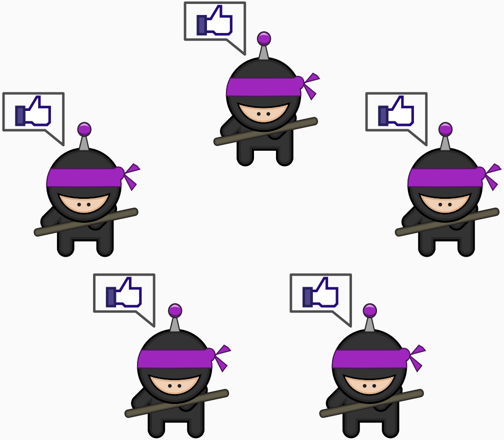

Sensei III's questions: Succinctness–An Example

Protocol for: Are there at least 2^ℓ sick ninjas?

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- $(2^\ell, n) \mapsto (2^\ell, 2^\ell)$
- Can be generalized to non-powers of 2

Succinctness

Just gave a protocol for $\mathbf{X \geq c}$ with $\mathcal{O}(\log c)$ states.

Succinctness

Just gave a protocol for **X ≥ c** with $\mathcal{O}(\log c)$ states.

Is $\mathcal{O}(\log \log c)$ possible?

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Not for every $\mathbf{c}$ ...

Blondin, E., Jaax STACS'18

There exist infinitely many $\mathbf{c}$ such that every protocol for $\mathbf{X} \geq \mathbf{c}$ has at least $(\log \mathbf{c})^{1/4}$ states

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There exist infinitely many $\mathbf{c}$ such that every protocol for $\mathbf{X} \geq \mathbf{c}$ has at least $(\log \mathbf{c})^{1/4}$ states

...but for some $\mathbf{c}$, if we allow leaders:

Blondin, E., Jaax STACS'18

For infinitely many $\mathbf{c}$ there is a protocol with two leaders and $\mathcal{O}(\log \log \mathbf{c})$ states that computes $\mathbf{X} \geq \mathbf{c}$

Succinctness

Blondin, E., Jaax STACS'18

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Proof:

Succinctness

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Proof:

- **Mayr and Meyer '82**: For every n there is a commutative semigroup presentation and two elements s, t such that the shortest word α leading from s to t (i.e., $t = s\,\alpha$) has length $|\alpha| \geq 2^{2^n}$

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- Construct a protocol that "simulates" derivations in the semigroup

Succinctness

$\mathcal{O}(\log \log c)$ without leaders?

Succinctness

$\mathcal{O}(\log \log c)$ without leaders? Open

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Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude ...

Checking correctness

Protocols can become complex, even for B ≥ R:

Fast and Exact Majority in Population Protocols

Dan Alistarh
Microsoft Research

Rati Gelashvili*
MIT

Milan Vojnović
Microsoft Research

$weight(x) = \begin{cases} |x| & \text{if } x \in StrongStates \text{ or } x \in WeakStates; \\ 1 & \text{if } x \in IntermediateStates. \end{cases}$

$sgn(x) = \begin{cases} 1 & \text{if } x \in \{+0, 1_d, \ldots, 1_1, 3, 5, \ldots, m\}; \\ -1 & \text{otherwise.} \end{cases}$

$value(x) = sgn(x) \cdot weight(x)$

```
/* Functions for rounding state interactions */
```

$\phi(x) = -1_1$ if $x = -1; 1_1$ if $x = 1; x,$ otherwise

$R_\downarrow(k) = \phi(k$ if k odd integer, $k-1$ if k even)

$R_\uparrow(k) = \phi(k$ if k odd integer, $k+1$ if k even)

$Shift\text{-}to\text{-}Zero(x) = \begin{cases} -1_{j+1} & \text{if } x = -1_j \text{ for some index } j < d \\ 1_{j+1} & \text{if } x = 1_j \text{ for some index } j < d \\ x & \text{otherwise.} \end{cases}$

$Sign\text{-}to\text{-}Zero(x) = \begin{cases} +0 & \text{if } sgn(x) > 0 \\ -0 & \text{oherwise.} \end{cases}$

procedure update$\langle x, y \rangle$
  if $(weight(x) > 0$ *and* $weight(y) > 1)$ ***or*** $(weight(y) > 0$ *and* $weight(x) > 1)$ **then**
    $x' \leftarrow R_\downarrow\left(\frac{value(x)+value(y)}{2}\right)$ and $y' \leftarrow R_\uparrow\left(\frac{value(x)+value(y)}{2}\right)$
  else if $weight(x) \cdot weight(y) = 0$ *and* $value(x) + value(y) > 0$ **then**
    if $weight(x) \neq 0$ **then** $x' \leftarrow Shift\text{-}to\text{-}Zero(x)$ and $y' \leftarrow Sign\text{-}to\text{-}Zero(x)$
    else $y' \leftarrow Shift\text{-}to\text{-}Zero(y)$ and $x' \leftarrow Sign\text{-}to\text{-}Zero(y)$
  else if $(x \in \{-1_d, +1_d\}$ *and* $weight(y) = 1$ *and* $sgn(x) \neq sgn(y))$ ***or***
      $(y \in \{-1_d, +1_d\}$ *and* $weight(x) = 1$ *and* $sgn(y) \neq sgn(x))$ **then**
    $x' \leftarrow -0$ and $y' \leftarrow +0$
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  else
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How can we verify correctness automatically?

Checking correctness—Early days

Model checkers:

- `PAT`: model checker with global fairness

  (Sun, Liu, Song Dong and Pang CAV'09)

- `bp-ver`: graph exploration

  (Chatzigiannakis, Michail and Spirakis SSS'10)

- Conversion to counter machines + `PRISM/Spin`

  (Clément, Delporte-Gallet, Fauconnier and Sighireanu ICDCS'11)

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Only for populations of fixed size!

Checking correctness—Early days

Theorem provers:

- Verification with the interactive theorem prover Coq

(Deng and Monin TASE'09)

Checking correctness—Early days

Theorem provers:

- Verification with the interactive theorem prover **Coq**

(Deng and Monin TASE'09)

Not automatic!

Checking correctness—Early days

Theorem provers:

- Verification with the interactive theorem prover **Coq**

(Deng and Monin TASE'09)

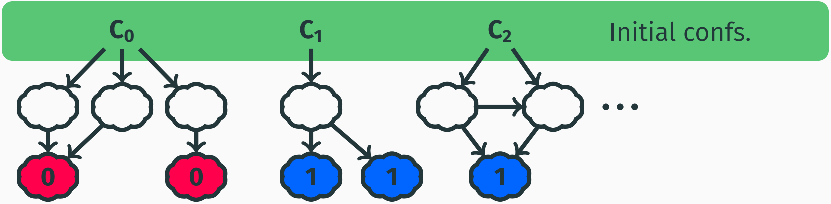
Challenge: verifying automatically

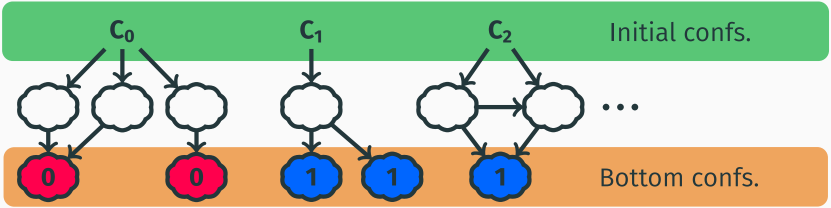
<u>all</u> sizes

E., Ganty, Leroux, Majumdar   Acta Inf.'17

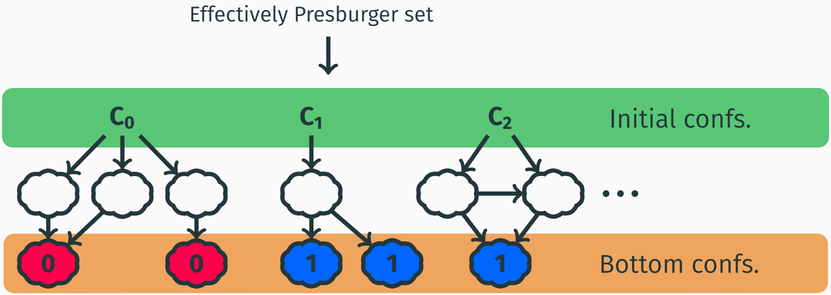
It is decidable if a population protocol is well specified (i.e., if it computes some predicate).

Checking correctness—Decidability





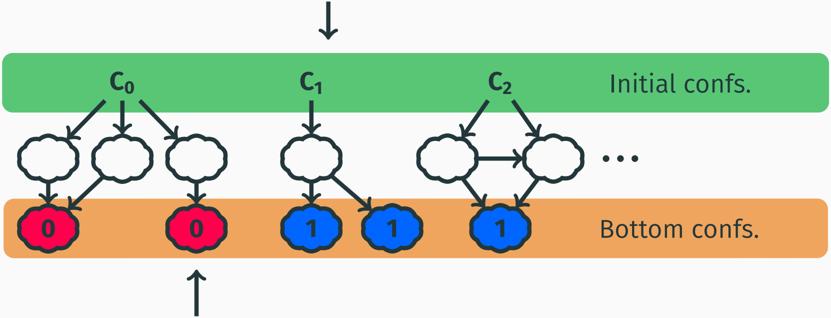
C_0
C_1
C_2
Initial confs.
...
0
0
1
1
1
Bottom confs.



Effectively Presburger set
C_0
C_1
C_2
Initial confs.
...
0
0
1
1
1
Bottom confs.

Checking correctness—Decidability

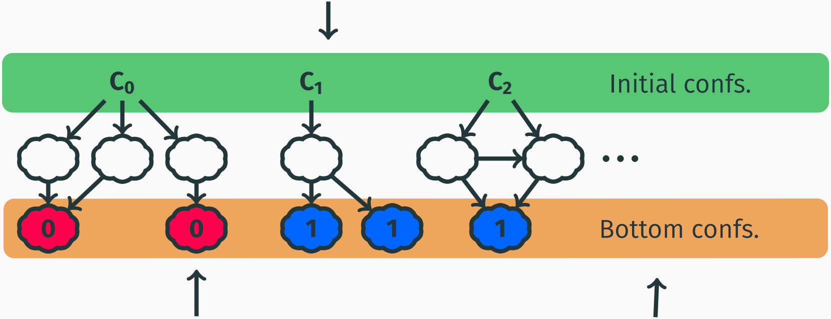
Effectively Presburger set



Eilenberg and Schützenberger '69:
Semilinear set
$\rightarrow$ Presburger

Checking correctness—Decidability

Effectively Presburger set



Eilenberg and Schützenberger '69:
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Leroux '11:
Effectively semilinear
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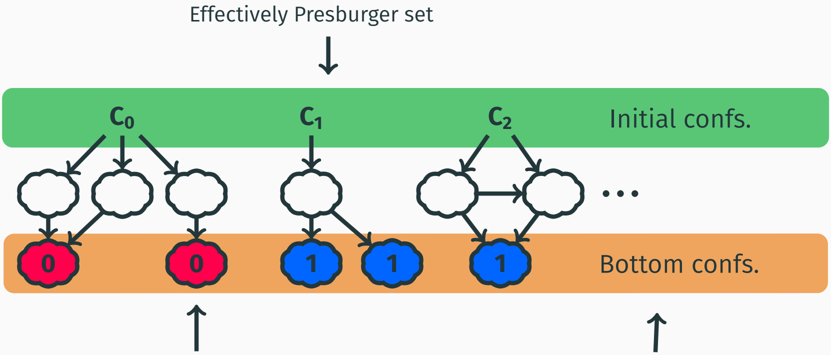
Checking correctness—Decidability



Eilenberg and Schützenberger '69:
Semilinear set
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Leroux '11:
Effectively semilinear
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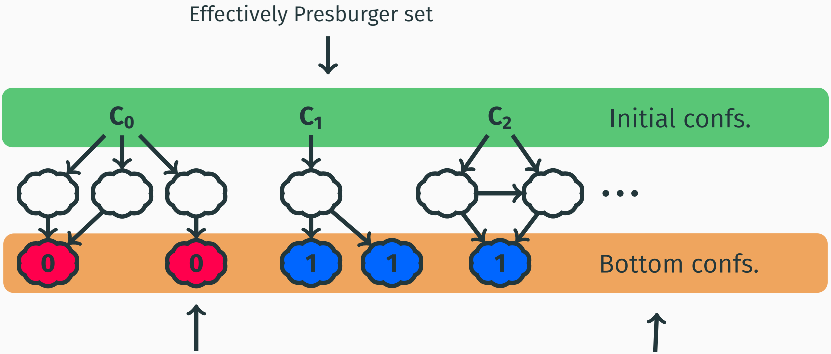
Reduction to the VAS reachability problem between Presburger sets

Checking correctness—Decidability



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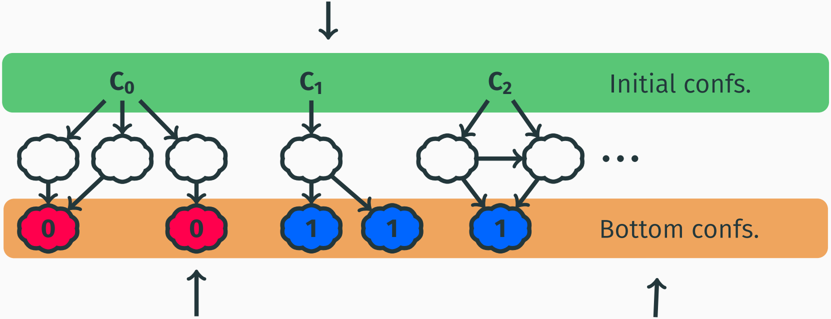
Reduction to the VAS reachability problem between Presburger sets
$\Rightarrow$ Reduction to the VAS reachability problem (VAS engineering)

Checking correctness—Decidability

Effectively Presburger set



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Reduction to the VAS reachability problem between Presburger sets

$\Rightarrow$ Reduction to the VAS reachability problem (VAS engineering)

$\Rightarrow$ Decidable (**Mayr '81, Kosaraju '83**).

E., Ganty, Leroux, Majumdar Acta Inf.'17

It is decidable if a population protocol computes a given predicate (Presburger formula).

E., Ganty, Leroux, Majumdar   Acta Inf.’17

It is decidable if a population protocol computes a given predicate (Presburger formula).

There is an algorithm that returns the predicate computed by a well-specified protocol.

E., Ganty, Leroux, Majumdar Acta Inf.'17

VAS reachability is reducible to the well-specification problem for population protocols

E., Ganty, Leroux, Majumdar Acta Inf.'17

VAS reachability is reducible to the well-specification problem for population protocols

$\Rightarrow$ Well specification is EXSPACE-hard, and all known algorithms for it have hyper-ackermannian complexity

A class $\mathcal{P}$ of protocols is complete if for every Presburger predicate φ some protocol in $\mathcal{P}$ computes φ

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Goal: Find a complete class of protocols verifiable in reasonable time

Blondin, E., Jaax, Meyer , PODC'17

The class of strongly silent protocols is complete, and its verification problem is in DP.

Checking correctness—Feasibility

Intel Core i7-4810MQ CPU and 16 GB of RAM.

Protocol	Predicate	$\|Q\|$	$\|T\|$	Time[s]
Majority[1]	$x \geq y$	4	4	0.1
Approx. Majority[2]	Not well-specified	3	4	0.1
Broadcast[3]	$x_1 \vee \ldots \vee x_n$	2	1	0.1
Threshold[4]	$\Sigma_i \alpha_i x_i < c$	76	2148	2375.9
Remainder[5]	$\Sigma_i \alpha_i x_i \bmod 70 = 1$	72	2555	3176.5
Sick ninjas[6]	$x \geq 50$	51	1275	181.6
Sick ninjas[7]	$x \geq 325$	326	649	3470.8
Poly-log sick ninjas	$x \geq 8 \cdot 10^4$	66	244	12.79

[1] Draief et al., 2012 [2] Angluin et al., 2007 [3] Clément et al., 2011
[4][5] Angluin et al., 2006 [6] Chatzigiannakis et al., 2010 [7] Clément et al., 2011

Blondin, E., Jaax, Meyer , PODC'17

The class of strongly silent protocols is complete, and its verification problem is in DP.

Mission accomplished?

Blondin, E., Jaax, Meyer , PODC'17

The class of strongly silent protocols is complete, and its verification problem is in DP.

Mission accomplished?

Not yet. For some predicates no strongly silent *succinct* protocols are known.

A class $\mathcal{P}$ of protocols is complete and succinct if for every Presburger predicate φ some protocol in $\mathcal{P}$ with $\log(|\varphi|)$ states computes φ

A class $\mathcal{P}$ of protocols is complete and efficient if for every Presburger predicate φ some protocol in $\mathcal{P}$ computes φ in $\mathcal{O}(n^2 \log n)$ time.

Checking correctness—Feasibility

Are strongly silent protocols complete and succinct?

Checking correctness—Feasibility

Are strongly silent protocols complete and succinct?

Are strongly silent protocols complete and efficient?

Checking correctness—Feasibility

Are strongly silent protocols complete and succinct?

Are strongly silent protocols complete and efficient?

What is the lowest expected time for a complete class of protocols?

Checking correctness—Feasibility

Are strongly silent protocols complete and succinct?

Are strongly silent protocols complete and efficient?

What is the lowest expected time for a complete class of protocols?

...and for a complete and succinct class?

Are strongly silent protocols complete and succinct?

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What is the lowest expected time for a complete class of protocols?

...and for a complete and succinct class?

...and for a complete and efficient class?

Are strongly silent protocols complete and succinct? Open

Are strongly silent protocols complete and efficient? Open

What is the lowest expected time for a complete class of protocols? Open

...and for a complete and succinct class? Open

...and for a complete and efficient class? Open

Sensei III's questions

What predicates can we compute?

How fast can we compute them?

How succinctly can we compute them?

How can I check correctness?

How can I check efficiency?

To conclude ...

Our approach:

- Most protocols are naturally designed in stages
- Construct these stages automatically
- Derive upper bounds on $Inter(n)$ from stages structure

Checking expected termination time

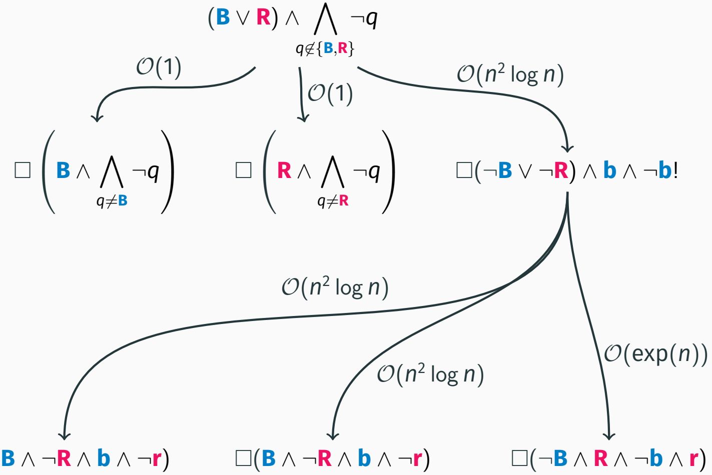
Blondin, E., Kucera CONCUR’18

$\mathbf{B}, \mathbf{R} \mapsto \mathbf{b}, \mathbf{b}$

$\mathbf{B}, \mathbf{r} \mapsto \mathbf{B}, \mathbf{b}$

$\mathbf{R}, \mathbf{b} \mapsto \mathbf{R}, \mathbf{r}$

$\mathbf{b}, \mathbf{r} \mapsto \mathbf{b}, \mathbf{b}$



Checking expected termination time

- Prototype implemented in python™ + Microsoft Z3

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- Can report: $\mathcal{O}(1), \mathcal{O}(n^2), \mathcal{O}(n^2 \log n), \mathcal{O}(n^3), \mathcal{O}(\text{poly}(n))$ or $\mathcal{O}(\exp(n))$

- Prototype implemented in python™ + Microsoft Z3
- Can report: $\mathcal{O}(1), \mathcal{O}(n^2), \mathcal{O}(n^2 \log n), \mathcal{O}(n^3), \mathcal{O}(\mathrm{poly}(n))$ or $\mathcal{O}(\exp(n))$
- Decidability of checking $Inter(n) \geq f(n)$?
  Open

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To conclude ...

Peregrine: a tool for population protocols Blondin, E., Jaax CAV'18

Peregrine: Haskell + Microsoft Z3 + JavaScript

`peregrine.model.in.tum.de`

- Design of protocols
- Manual and automatic simulation
- Statistics of properties such as termination time
- Automatic verification of correctness
- More to come!

Conclusion

Population protocols are a great model to study fundamental questions of distributed computation:

- Power of anonymous computation
- Network-independent algorithms
- Role of leaders
- Emergent behaviour and its limits

Conclusion

…and of formal verification:

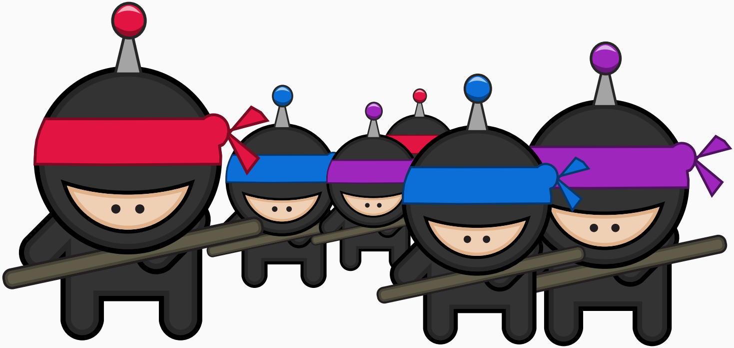
- Verification of stochastic parameterized systems (parameterization, liveness under fairness)
- Automatic synthesis of parameterized systems

Join the team!

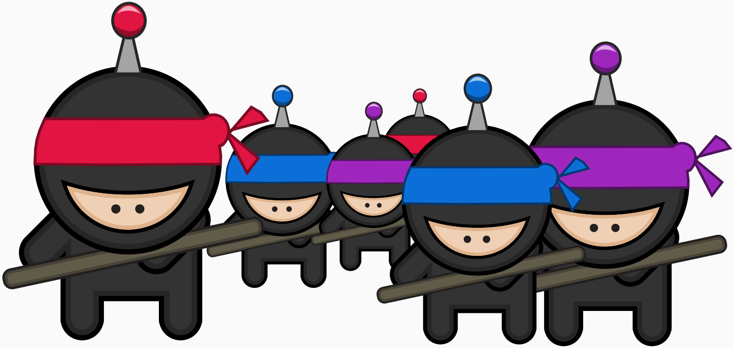
ERC Advanced Grant —

PaVeS: Parameterized Verification and Synthesis

- Goal: Develop proof and synthesis techniques for distributed algorithms working correctly for an arbitrary number of processes
- Start of the project: Sept. 1, 2018
- Start of employment: flexible, from Sept. 1, 2018 to about Sept. 1, 2019



THANK YOU!



Go!

THANK YOU!