

Computing Least Fixed Points of Probabilistic Systems of Polynomials

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Javier Esparza¹, Andreas Gaiser¹, Stefan Kiefer²

¹Technische Universität München

²Oxford University

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Outline

- 1 What is a Probabilistic System of Polynomials (PSP)?
- 2 Why studying PSPs?
- 3 Algorithms
 - An Exact Algorithm for Consistency
 - An Exact Algorithm for Lower and Upper Bounds of $LFP(f)$
- 4 Case study: PSPs in Physics

Definition of a PSP

- We investigate polynomial equation systems

$$X_1 = f_1(X_1, \dots, X_n)$$

...

$$X_n = f_n(X_1, \dots, X_n)$$

where the f_i are polynomials over X_1, \dots, X_n .

- Important restriction: The coefficients of each f_i are **nonnegative** and **sum up to 1**.
- The vector $f := (f_1, \dots, f_n)^\top$ is called a probabilistic system of polynomials (PSP).

An Example

2-dimensional PSP

$$\begin{aligned}X_1 &= \frac{4}{5}X_1X_2 + \frac{1}{5} \\X_2 &= \frac{2}{5}X_1X_1 + \frac{1}{10}X_2 + \frac{1}{2}\end{aligned}$$

leads to the PSP $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ with

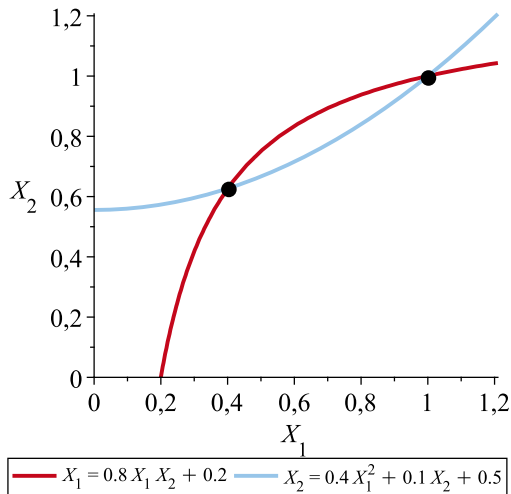
$$\begin{aligned}f_1(X_1, X_2) &= \frac{4}{5}X_1X_2 + \frac{1}{5} \\f_2(X_1, X_2) &= \frac{2}{5}X_1X_1 + \frac{1}{10}X_2 + \frac{1}{2}.\end{aligned}$$

Fixed Points

- For every PSP, $\bar{1} = (1, \dots, 1)$ is a fixed point.
- We are interested in the **least nonnegative fixed point (LFP)** of f , where we mean “least” with respect to the order “ \leq ” defined componentwise.

An Example

$$X_1 = \frac{4}{5}X_1X_2 + \frac{1}{5}$$
$$X_2 = \frac{2}{5}X_1X_1 + \frac{1}{10}X_2 + \frac{1}{2}$$



Problems

Problem 1 (Consistency problem)

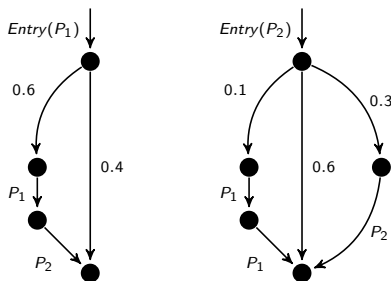
Given a PSP f , decide whether $\text{LFP}(f) = \bar{\mathbf{I}}$.

Problem 2 (Computing Lower and Upper bounds)

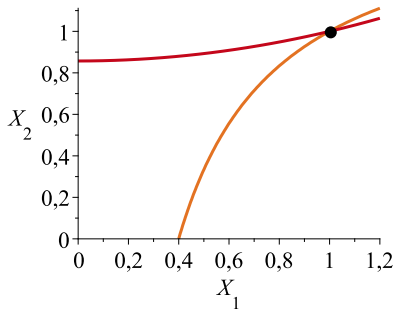
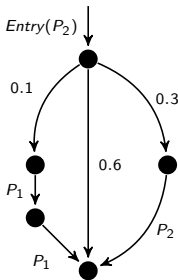
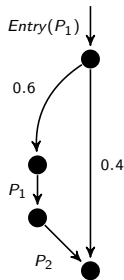
Given a PSP f , for a given $\epsilon > 0$, compute \mathbf{lb} , \mathbf{ub} such that $\mathbf{lb} \leq \text{LFP}(f) \leq \mathbf{ub}$ with $\mathbf{ub} - \mathbf{lb} \leq \bar{\epsilon}$.

- If $\text{LFP}(f) = \bar{\mathbf{I}}$ then f is **consistent**, otherwise **inconsistent**.
- Why are those problems interesting?

Termination probability of probabilistic recursive programs



- Probabilistic flow graphs of two simple procedures P_1 and P_2 .
- **Termination probability** for $P_i =$ Probability that a call $P_i()$ eventually terminates.



Corresponding equation system

$$X_1 = 0.6X_1X_2 + 0.4$$

$$X_2 = 0.1X_1X_2 + 0.3X_2 + 0.6$$

- Termination probabilities = LFP of the corresponding PSP.
- Here: $LFP(f) = (1, 1)$
 \Rightarrow Termination with probability 1.
- Termination with prob. 1 depends not only on the program structure.

Applications of PSPs: Multi-type branching processes

$$X_1 \xrightarrow{0.6} \{X_1, X_2\}$$

$$X_1 \xrightarrow{0.4} \{\}$$

$$X_2 \xrightarrow{0.1} \{X_1, X_1\}$$

$$X_2 \xrightarrow{0.3} \{X_2\}$$

$$X_2 \xrightarrow{0.6} \{\}$$

Applications in various areas:

- **Verification of probabilistic programs:** Termination probability of Probabilistic Pushdown Systems and Recursive Markov Chains
- **Biology:** Reproduction and extinction of species
- **Natural Language processing:** Stochastic context-free grammars
- **Physics:** See case study at the end

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Consistency can be decided in weakly polynomial time.

Problem 1 (Consistency problem)

Given a PSP f , decide whether $\text{LFP}(f) = \bar{\mathbf{1}}$.

- This can be decided [Etessami/Yannakakis, 2009] in (weakly) polynomial time by checking whether the following LP problem has a solution:

$$f'(\bar{\mathbf{1}})\mathbf{x} \geq (1 + 2^{-c_f})\mathbf{x} \text{ with } \mathbf{x} \geq \bar{\mathbf{0}} \text{ and } \sum_{i=1}^n x_i = 1.$$

- Problem: c_f , although polynomial in f , can be very large...

An almost consistent family of PSPs

- a family of inconsistent (but “almost consistent”) PSPs:

$$X_1 = 0.5X_1^2 + 0.1X_n^2 + 0.4$$

$$X_2 = 0.01X_1^2 + 0.5X_2 + 0.49$$

...

$$X_n = 0.01X_{n-1}^2 + 0.5X_n + 0.49.$$

- **Inexact** LP-solvers cannot handle the instances with $n > 10$.
- Experiments with Maple's **exact** Simplex package

	$n = 100$	$n = 200$	$n = 400$	$n = 600$	$n = 1000$
Exact LP	2 sec	8 sec	67 sec	208 sec	> 2h

Our new consistency-check algorithm

- Algorithm for consistency of **strongly connected** PSPs:
 - 1 Solve the system $(Id - f'(\bar{\mathbf{I}}))\mathbf{v} = \bar{\mathbf{0}}$.
 - 2 If a solution $\mathbf{v} \neq \bar{\mathbf{0}}$ exists, return true iff $\mathbf{v} \succ \bar{\mathbf{0}}$ or $\mathbf{v} \prec \bar{\mathbf{0}}$.
 - 3 Else find the unique solution of the system $(Id - f'(\bar{\mathbf{I}}))\mathbf{v} = \bar{\mathbf{1}}$.
 - 4 If $\mathbf{v} \geq \bar{\mathbf{1}}$ and $f'(\bar{\mathbf{I}})\mathbf{v} < \mathbf{v}$ return true, else return false.

⇒ It suffices to solve two linear equation systems.

- We can generalize the algorithm easily to **arbitrary** PSPs.

Assessment

- No need for invoking Linear Programming
- The algorithm is strongly polynomial and very easy to implement.
- Comparison on the “almost consistent” family:

	$n = 100$	$n = 200$	$n = 400$	$n = 600$	$n = 1000$
Exact LP	2 sec	8 sec	67 sec	208 sec	> 2h
New alg.	1 sec	1 sec	4 sec	10 sec	29 sec

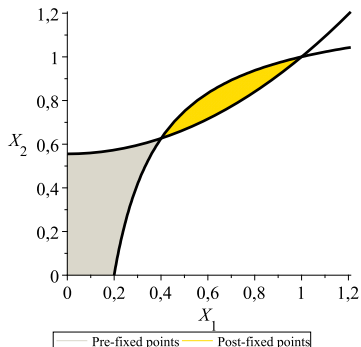
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Problem 2 (Computing Lower and Upper Bounds)

Given a PSP f , for a given $\epsilon > 0$, compute \mathbf{lb}, \mathbf{ub} such that
 $\mathbf{lb} \leq LFP(f) \leq \mathbf{ub}$ with $\mathbf{ub} - \mathbf{lb} \leq \bar{\epsilon}$.

Pre-fixed and Post-fixed Points

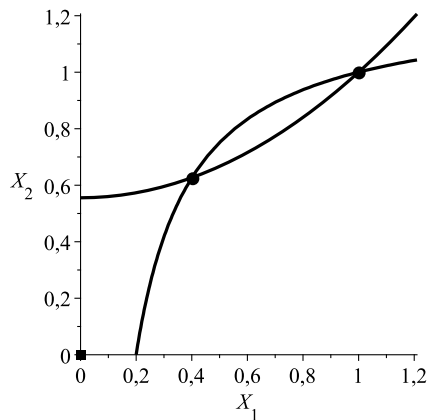
- $\mathbf{x} \in \mathbb{R}^n$ is a pre-fixed (post-fixed) point if $f(\mathbf{x}) \geq \mathbf{x}$ ($f(\mathbf{x}) \leq \mathbf{x}$).
- If \mathbf{x} is strictly greater than \mathbf{y} in all components we write $\mathbf{x} \succ \mathbf{y}$.
- $\mathbf{x} \in \mathbb{R}^n$ is a **strict** pre-fixed (post-fixed) point if $f(\mathbf{x}) \succ \mathbf{x}$ ($f(\mathbf{x}) \prec \mathbf{x}$).
- Pre-fixed points are lower bounds,
post-fixed points are upper bounds for $\text{LFP}(f)$.



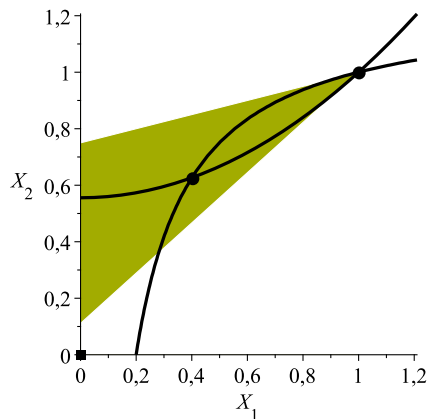
Obtaining lower bounds: Newton's method

- $\bar{0}, f(\bar{0}), f^2(\bar{0}), \dots$ are all pre-fixed points.
The sequence converges (in general slowly) to $\text{LFP}(f)$.
- ⇒ Apply **Newton's method** for finding zeros of the map $f(\mathbf{X}) - \mathbf{X}$
- Applying Newton to an approximation \mathbf{x} gives a better approximation $\mathcal{N}_f(\mathbf{x})$.
- $\bar{0}, \mathcal{N}_f(\bar{0}), \mathcal{N}_f(\mathcal{N}_f(\bar{0})), \dots$ **converges linearly** to $\text{LFP}(f)$ from below [Esparza, K., Luttenberger, 2010]
- But what about **upper bounds**? (Newton cannot be used for that)

Upper bounds

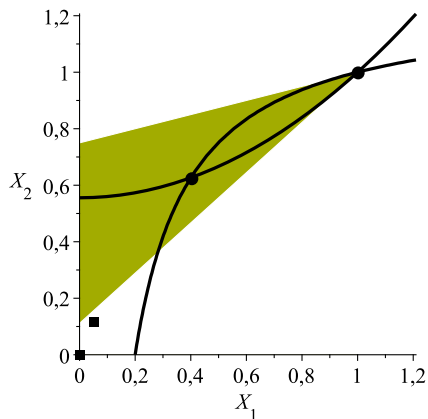


Upper bounds



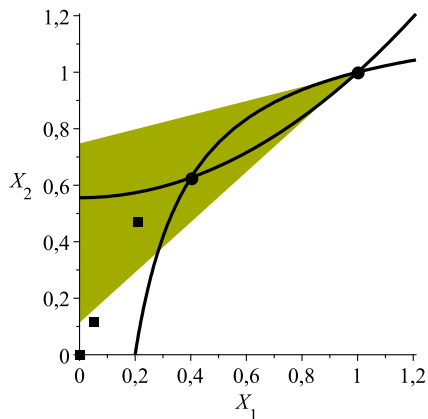
- $\text{LFP}(f) \neq \bar{1} \Rightarrow$ there exists a green area of points \mathbf{x} with $f'(\bar{1})(\bar{1} - \mathbf{x}) \succ (\bar{1} - \mathbf{x})$.

Upper bounds



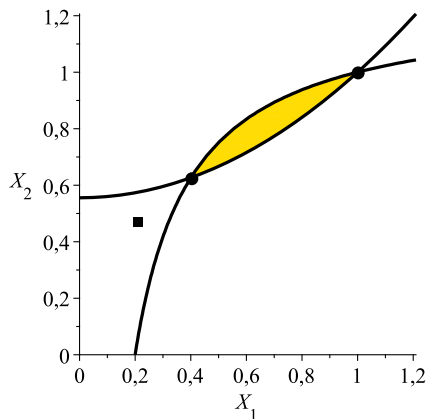
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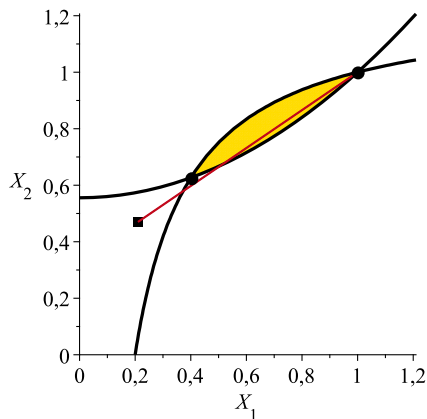
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- Any sequence $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \dots$ of points converging to $\text{LFP}(f)$ (e.g. Newton iterates) enters the green area.

Upper bounds



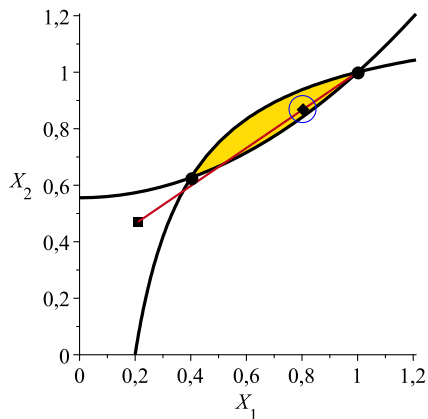
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- Using such a point we compute a strict **post-fixed point** \mathbf{p} .

Upper bounds



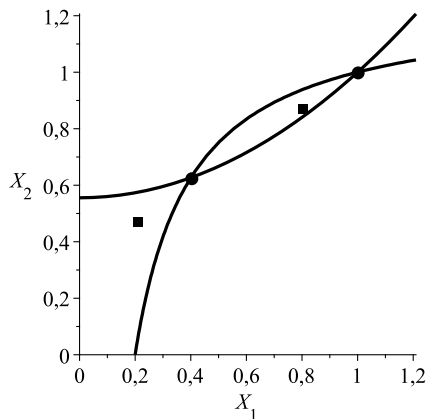
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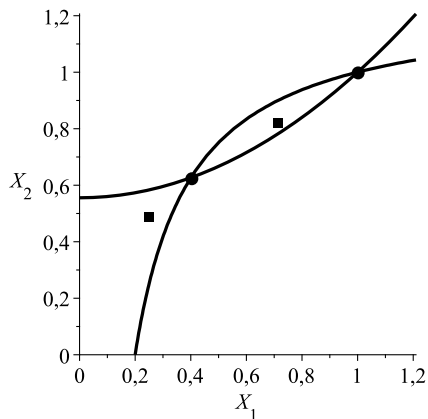
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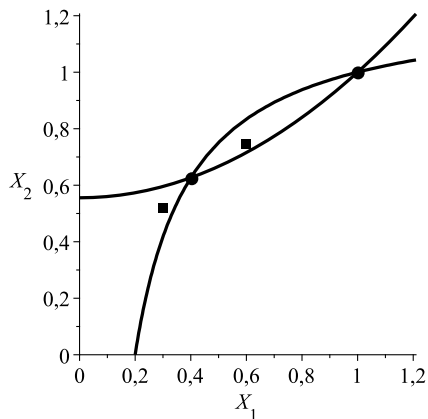
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- $\mathbf{p}, f(\mathbf{p}), f(f(\mathbf{p})), \dots$ converges linearly to $\text{LFP}(f)$ from above.

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The algorithm so far

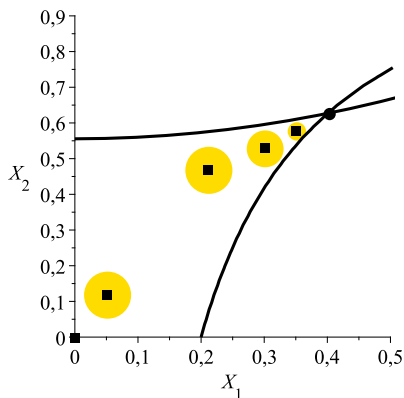
- 1 Set $\mathbf{lb} := \bar{0}$, $\mathbf{ub} := \bar{1}$.
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compute strict **post-fixed point \mathbf{p}** and set $\mathbf{ub} := \mathbf{p}$.
- 4 If $\mathbf{ub} \neq \bar{1}$, set $\mathbf{ub} := f(\mathbf{ub})$.
- 5 If $\mathbf{ub} - \mathbf{lb} \not\leq \bar{\epsilon}$ go to (2).

Problems with exact computations

- For computing a Newton iterate $\mathcal{N}_f(\mathbf{x})$ we have to solve a linear equation system.
- For reliable results: Exact (rational) arithmetic
- The number of bits needed to represent the exact iterates **grows exponentially** with the number of iterations.
- Similar problem with exact upper bounds.
- We want to use “inexact” arithmetic operations with finite precision, e.g. **floating-point arithmetic**, in a “controlled” and “local” fashion...
- ... Especially: Detection and correction of round-off errors

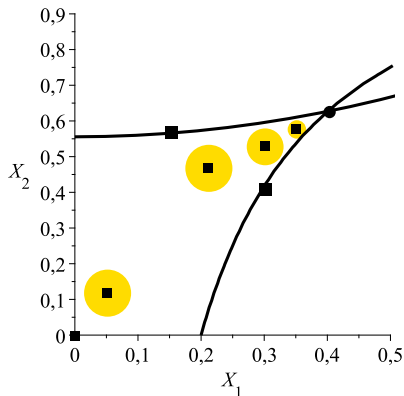
Avoiding exact computations

- Each Newton iterate $\mathcal{N}_f(\mathbf{x})$ is surrounded by a ϵ -ball C_x of points \mathbf{y} with $\bar{1} \succ f(\mathbf{y}) \succ \mathbf{y} \succ f(\mathbf{x})$.



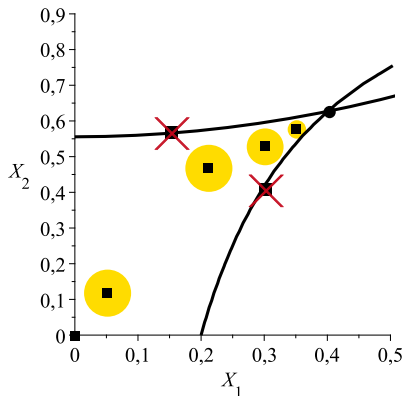
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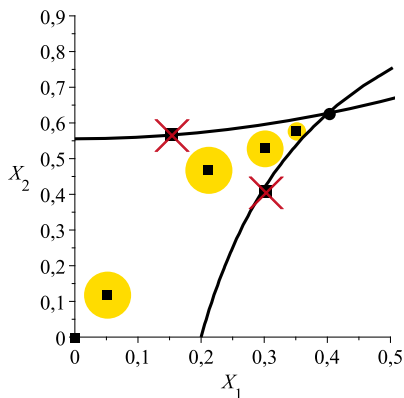
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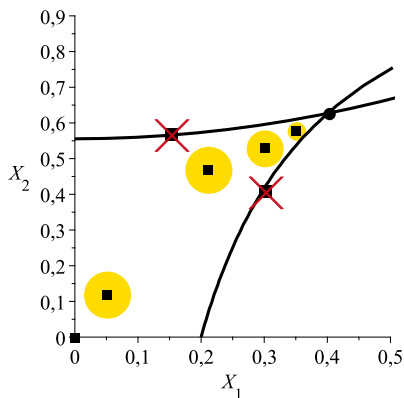
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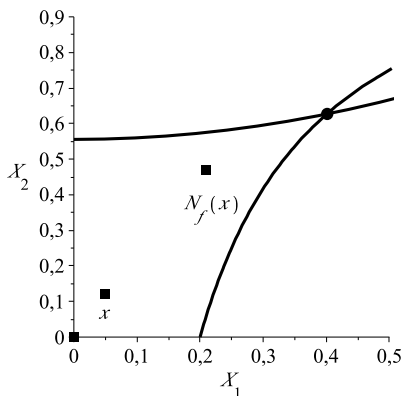
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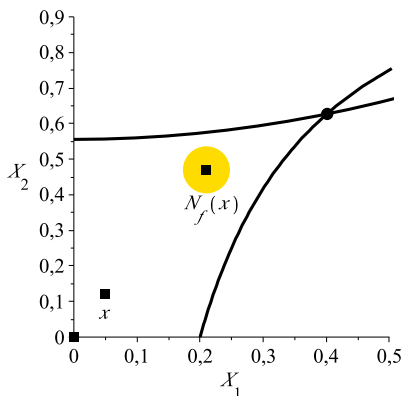
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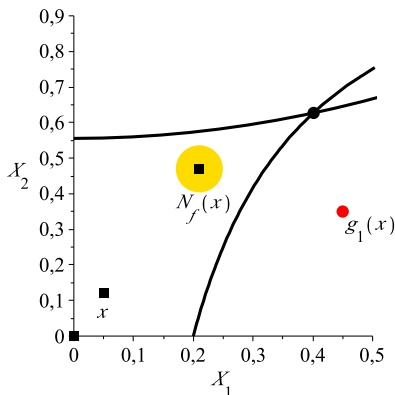
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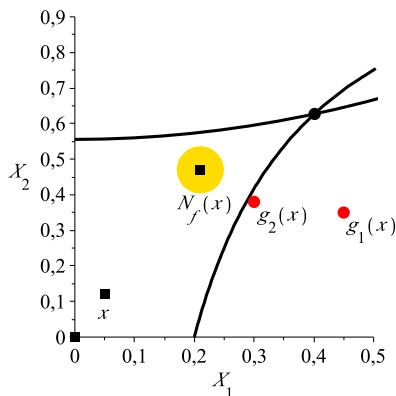
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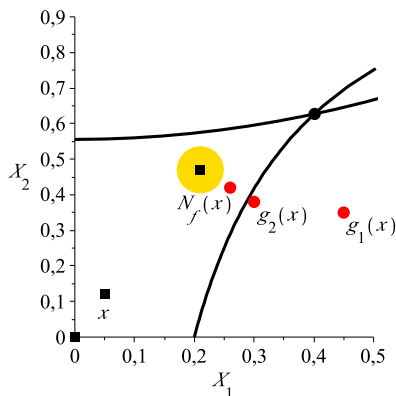
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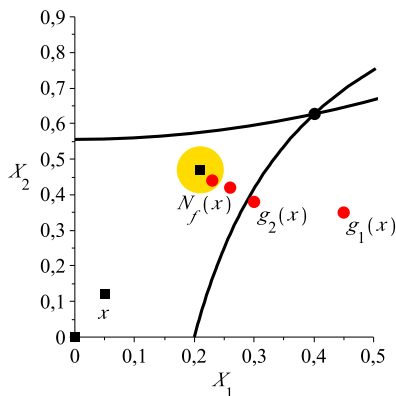
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The Floating Assignment

- We write

$$\mathbf{y} \leftarrow \mathcal{N}_f(\mathbf{x}) \text{ such that } \bar{\mathbf{I}} \succ f(\mathbf{y}) \succ \mathbf{y} \succ f(\mathbf{x})$$

as syntactic sugar for

- 1 Set $i := 0$.
 - 2 Compute $\mathbf{y} := g^{(i)}(\mathbf{x})$.
 - 3 If not $\bar{\mathbf{I}} \succ f(\mathbf{y}) \succ \mathbf{y} \succ f(\mathbf{x})$ set $i := i + 1$ and go to (2).
- Computation of $g^{(i)}$: E.g. convert \mathbf{x} to a floating-point number, perform the operation, convert the result back.
 - $g^{(0)}(\mathbf{x}), g^{(1)}(\mathbf{x}), g^{(2)}(\mathbf{x}), \dots \rightarrow \mathcal{N}_f(\mathbf{x})$
 - Precision of the numbers/operations increases with i .
(Maple, GNU Multi-Precision Library)
 - We replace the computation of the iterates by floating assignments.

The Algorithm

- 1 Set $\mathbf{lb} := \bar{\mathbf{0}}, \mathbf{ub} := \bar{\mathbf{1}}$.
 - 2 Set $\mathbf{y} \leftarrow \mathcal{N}_f(\mathbf{x})$ such that $\bar{\mathbf{1}} \succ f(\mathbf{y}) \succ \mathbf{y} \succ f(\mathbf{x})$.
 - 3 If $f'(\bar{\mathbf{1}})(\bar{\mathbf{1}} - \mathbf{lb}) \succ (\bar{\mathbf{1}} - \mathbf{lb})$ and $\mathbf{ub} = \bar{\mathbf{1}}$,
compute strict post-fixed point \mathbf{p} and set $\mathbf{ub} := \mathbf{p}$.
 - 4 If $\mathbf{ub} \neq \bar{\mathbf{1}}$, set $\mathbf{ub} := f(\mathbf{ub})$.
 - 5 If $\mathbf{ub} - \mathbf{lb} \not\leq \bar{\epsilon}$ go to (2).
- Floating-Assignments also for upper bounds possible.

Summary

- The algorithm
 - ▶ computes **reliable lower and upper bounds** for $\text{LFP}(f)$, which are arbitrarily close.
 - ▶ uses **inexact arithmetic** for costly computations
 - ⇒ In practice, the precision needs to be increased only rarely.
 - ⇒ We observe a significant speed-up.

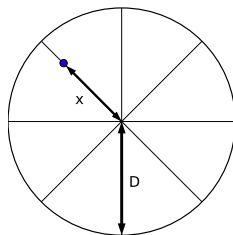
- 1 What is a Probabilistic System of Polynomials (PSP)?
- 2 Why studying PSPs?
- 3 Algorithms
 - An Exact Algorithm for Consistency
 - An Exact Algorithm for Lower and Upper Bounds of $LFP(f)$
- 4 Case study: PSPs in Physics



Explosion risk

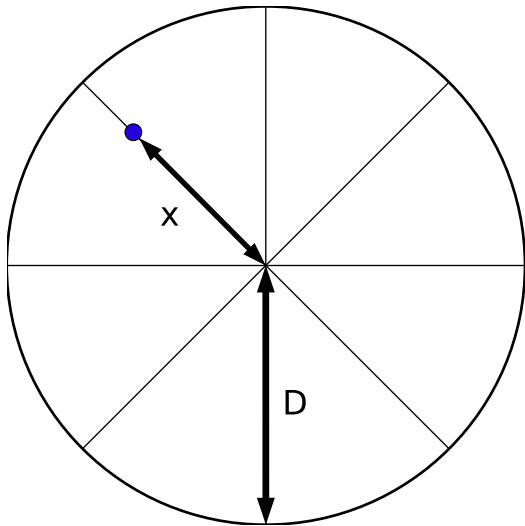
Nuclear Fission

- Example from [Harris, 1963].
- Single neutron with distance x to the centre of a ball of radius D of radioactive material.
- Before exiting the ball, the neutron might collide with a nucleus.
- Other free neutrons may emerge from the collision, which may trigger more collisions \Rightarrow Danger (or chance?) of chain reaction!



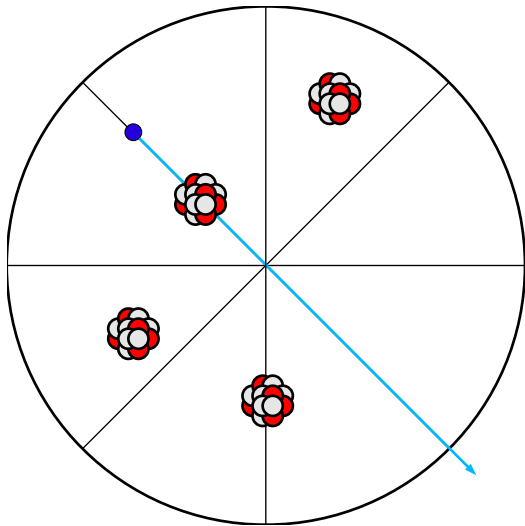
- In case of a collision, with probability ...

- 1 0.025, the neutron will be absorbed into the nucleus.
- 2 0.830, the neutron will be deflected by the nucleus.
- 3 0.070, the nucleus will break up and **two** neutrons emerge.
- 4 0.050, nucleus breaks and **3** new neutrons.
- 5 0.025, nucleus breaks and **4** new neutrons.



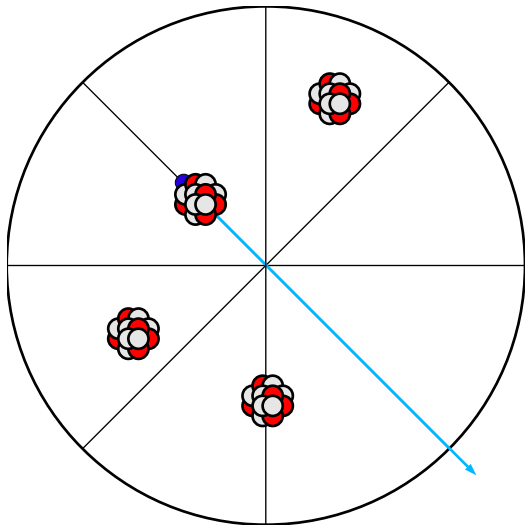
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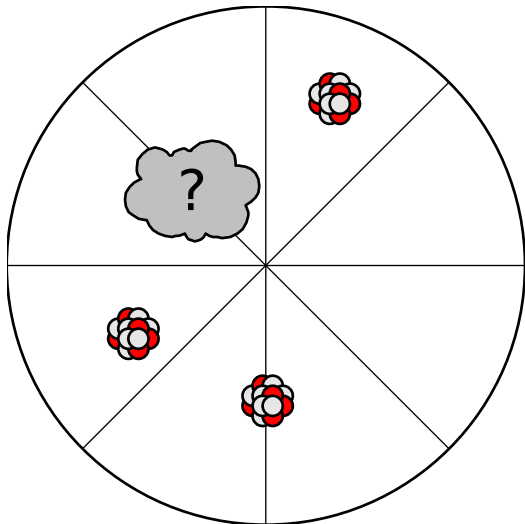
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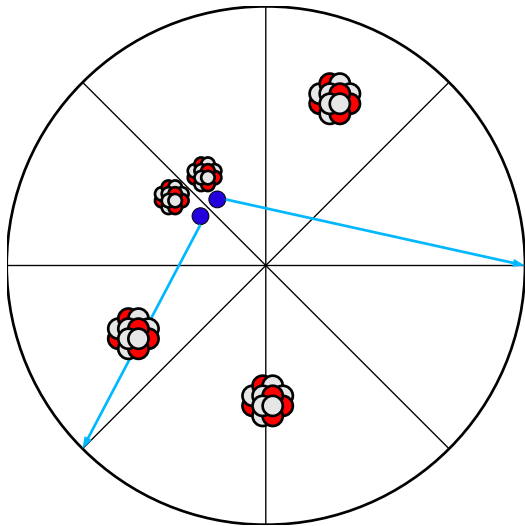
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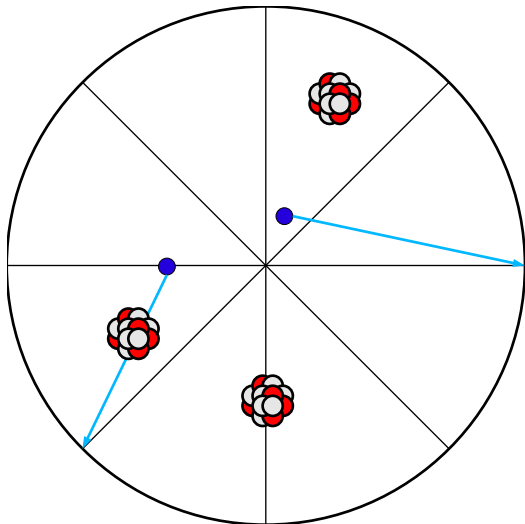
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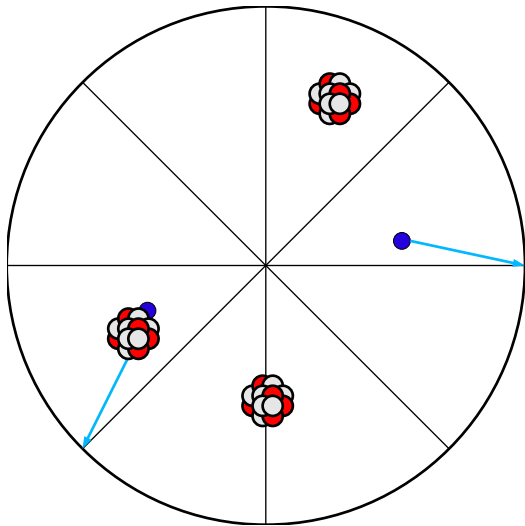
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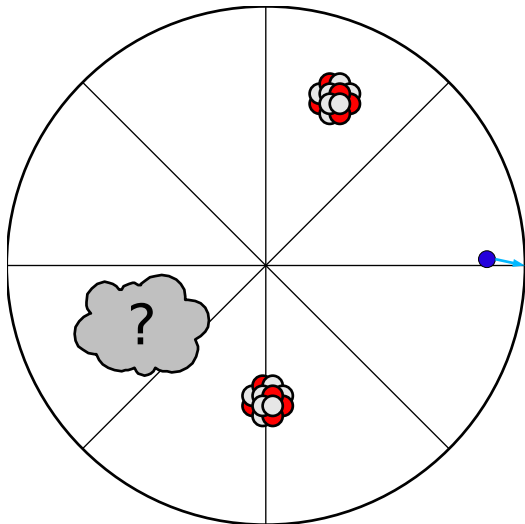
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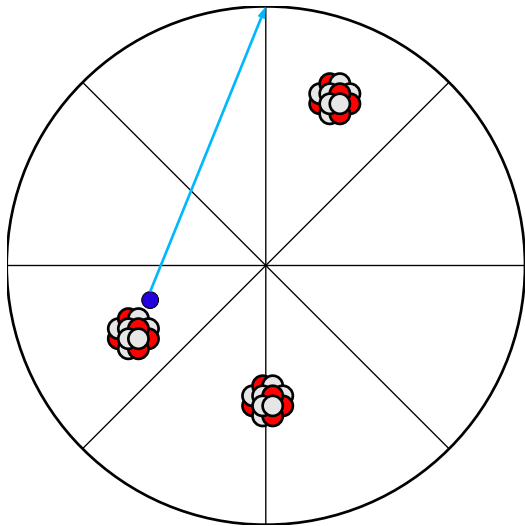
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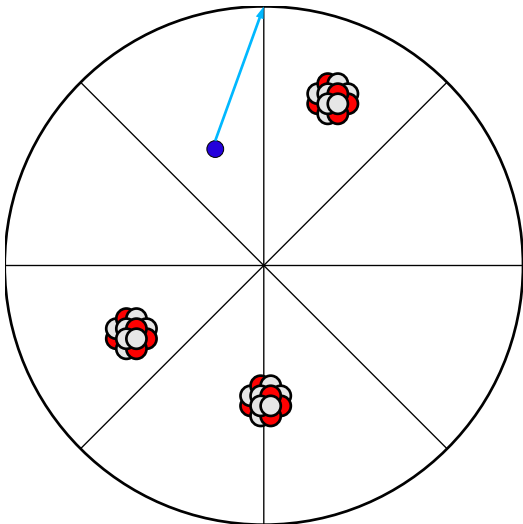
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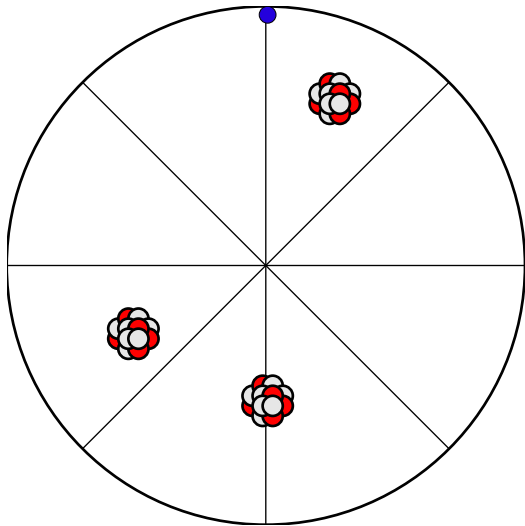
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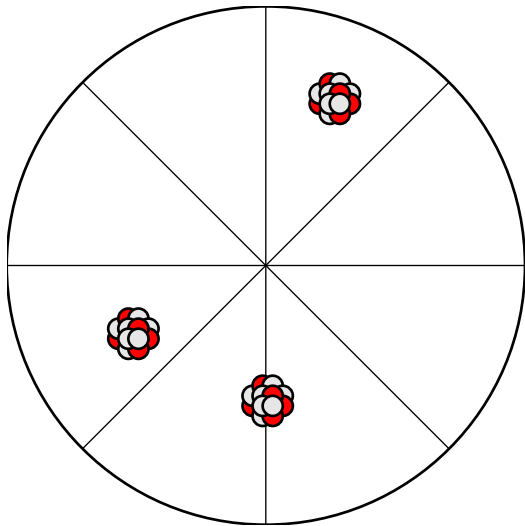
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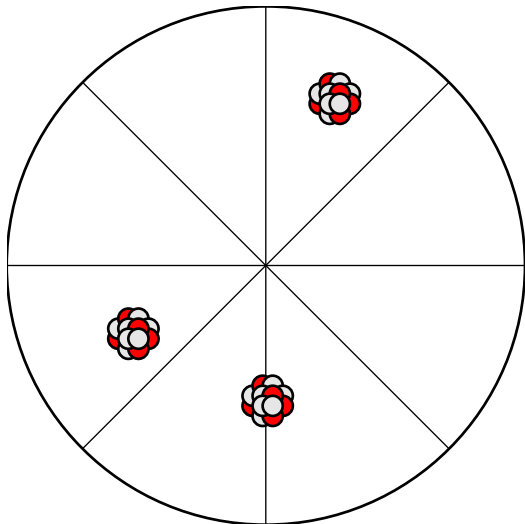
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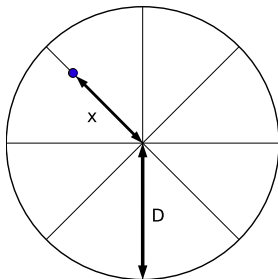
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- Given is $\ell(x)$: the probability that a neutron starting at x **leaves** the ball without collision.
- Given is $R(x, y)$: the probability that a neutron starting at x **collides** with a nucleus at y .

Wanted Values

- Wanted: $Q_D(x)$, the probability that from a single neutron, starting at x , only finitely many free neutrons emerge
= Probability of NO EXPLOSION
- Wanted: **Critical radius**, i.e., the largest D with $Q_D(0) = 1$



Discretization gives a PSP

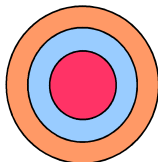
- $Q_D(x)$ satisfies the functional equation

$$Q_D(x) = \ell(x) + \int_0^D R(x, y) f(Q_D(y)) dy.$$

with $f(z) = 0.025 + 0.83z + 0.07z^2 + 0.05z^3 + 0.025z^4$.

- We discretize the interval $[0, D]$ into n shells with thickness D/n .

⇒ PSP with n variables X_1, \dots, X_n .



Discretization gives a PSP

- Resulting equation system
(constants $l_1, r_{i,j}$ can be numerically computed):

$$X_1 = l_1 + \sum_{i=1}^n r_{1,i} \cdot (0.025 + 0.83X_i + 0.07X_i^2 + 0.05X_i^3 + 0.025X_i^4)$$

$$X_2 = l_2 + \sum_{i=1}^n r_{2,i} \cdot (0.025 + 0.83X_i + 0.07X_i^2 + 0.05X_i^3 + 0.025X_i^4)$$

...

$$X_n = l_n + \sum_{i=1}^n r_{n,i} \cdot (0.025 + 0.83X_i + 0.07X_i^2 + 0.05X_i^3 + 0.025X_i^4)$$

Experiments

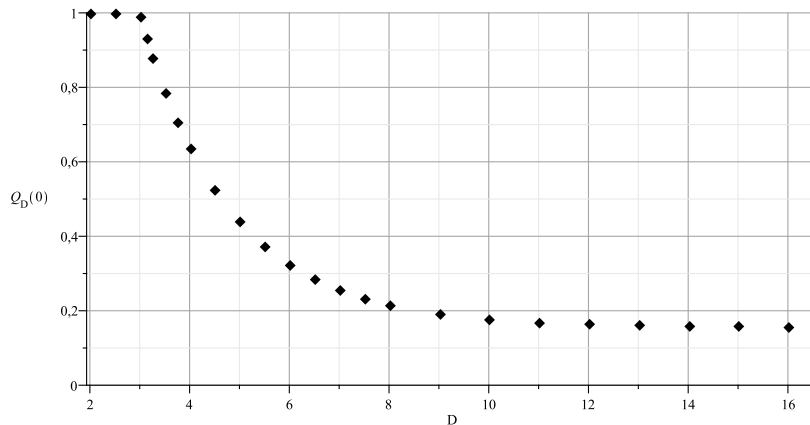
- Computation for $n = 100$, different radii D

D	2	3	6	10
Still safe?	✓	☠	☠	☠
Cons. check (our algorithm)	2s	2s	2s	2s
Cons. check (exact LP)	258s	124s	168s	222s
Approx. Q_D ($\epsilon = 0.001$)	4s	32s	21s	17s

- Numerical results are similar to [Harris, 1963].
- We observed at most two increases of precision per computation.

Experiments

- Values of $Q_D(0)$ for different radii D
- Binary search using consistency algorithm: Critical radius lies in $[2.981, 2.991]$ (Harris: ca. 2.9).



Thank you!



Theodore E. Harris

The theory of branching processes

Springer-Verlag, 1963



Kousha Etesami and Mihalis Yannakakis

Recursive Markov chains, stochastic grammars, and monotone systems of nonlinear equations

Journal of the ACM, 56(1):1-66, 2009



Javier Esparza, Andreas Gaiser, Stefan Kiefer

Computing least fixed points of probabilistic systems of polynomials
STACS 2010



Javier Esparza, Stefan Kiefer, Michael Luttenberger

Computing the least fixed point of positive polynomial systems
SIAM Journal on Computing, 39(6):2282-2335, 2010