

On Stabilization in Herman's Algorithm

ICALP 2011

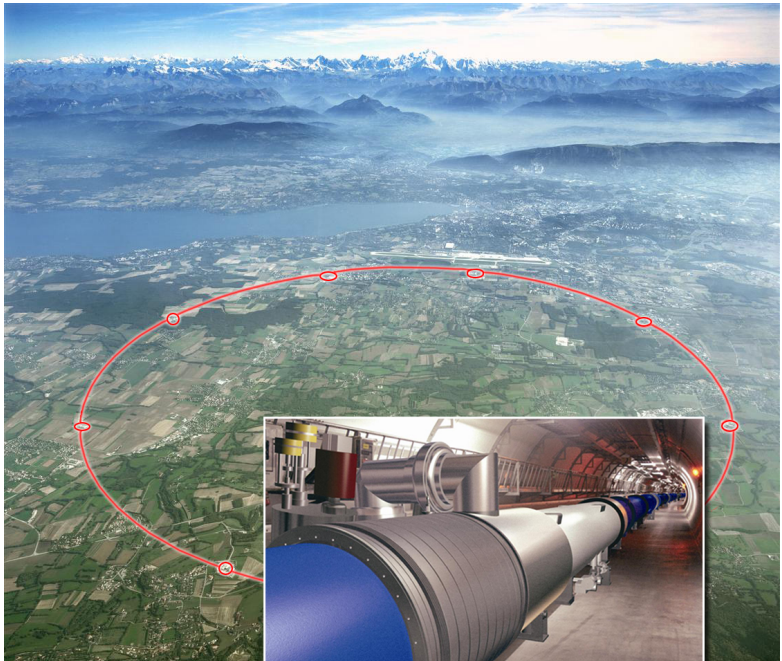
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James Worrell¹ Lijun Zhang³

¹Department of Computer Science, University of Oxford, UK

²Department of Computer Science, University of Leicester, UK

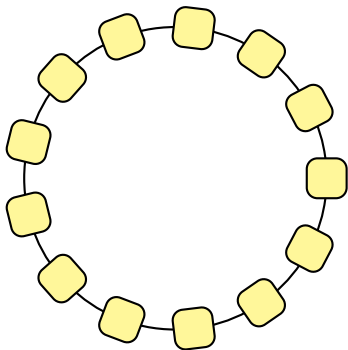
³DTU Informatics, Technical University of Denmark, Denmark

Particle Collision: CERN



Herman's Algorithm

Herman's algorithm is a randomized **leader election** protocol.

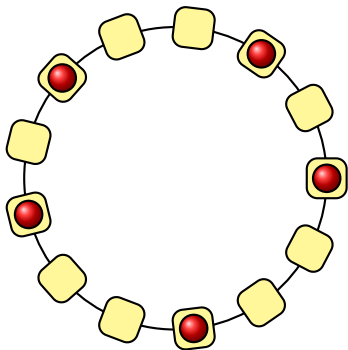


- Ring of N processors (N is odd)
- Some processors have a token.
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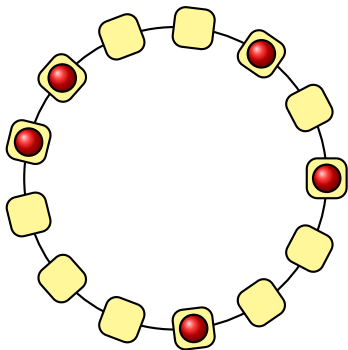


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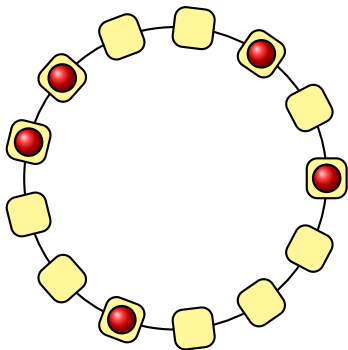


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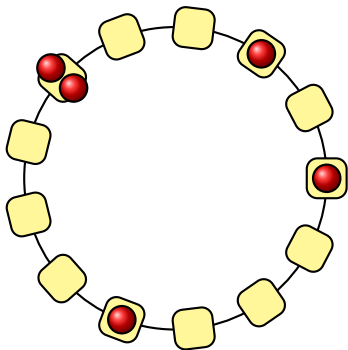


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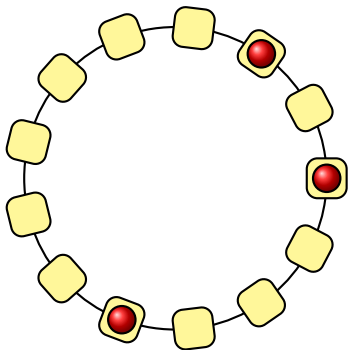


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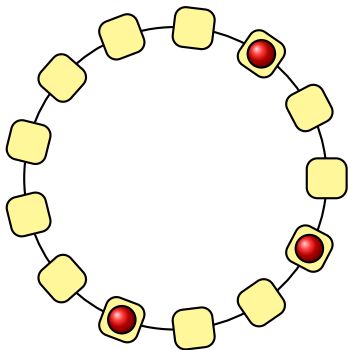


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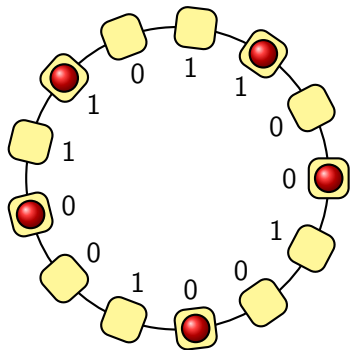


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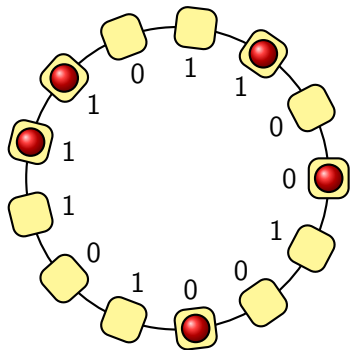
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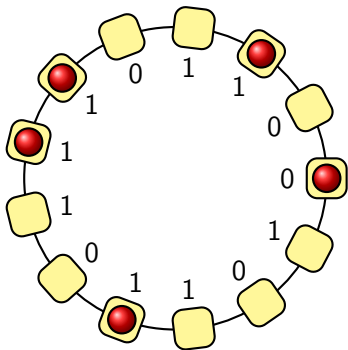
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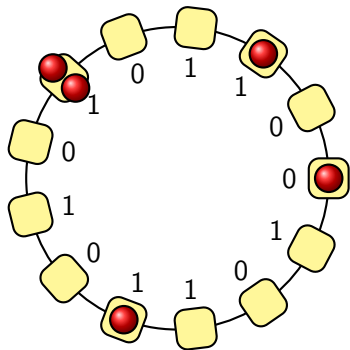
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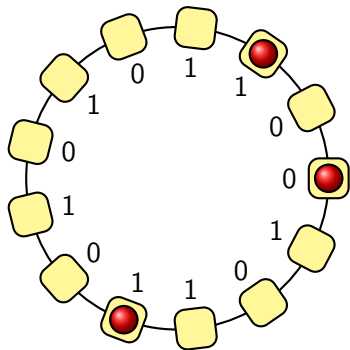
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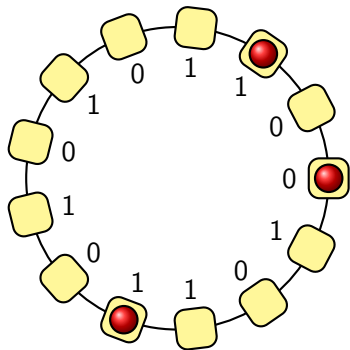
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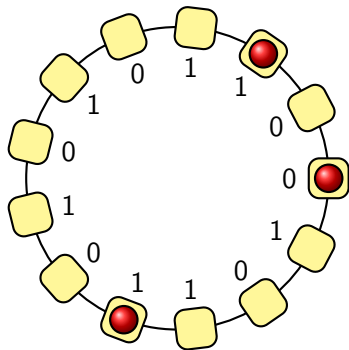
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Main Question:

How long does it take – on average – until a leader is elected?

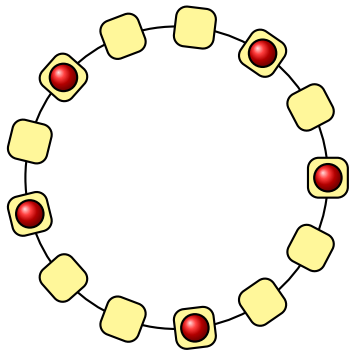
Herman's Algorithm

Two versions of Herman's algorithm:

- **Asynchronous:** Each processor with a token passes the token to its clockwise neighbour with rate λ
 - \Rightarrow continuous-time Markov chain
 - \Rightarrow (as seen on the previous slides)
- **Synchronous:** Each processor with a token
 - with probability $1/2$: passes it (i.e., flips its bit)
 - with probability $1/2$: keeps it (i.e., keeps its bit)
 - \Rightarrow discrete-time Markov chain

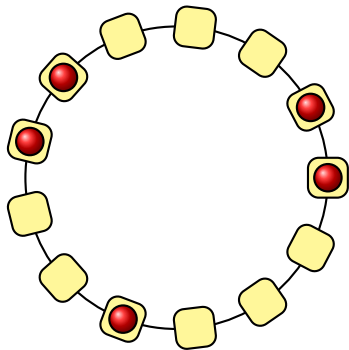
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Illustration of the synchronous version:



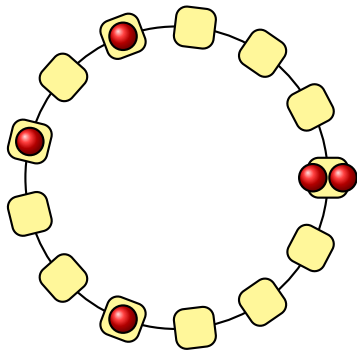
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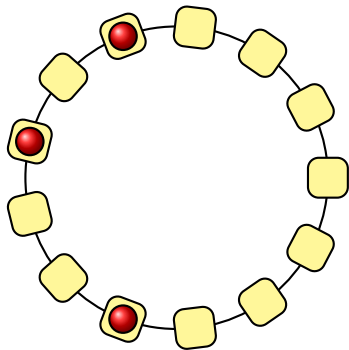
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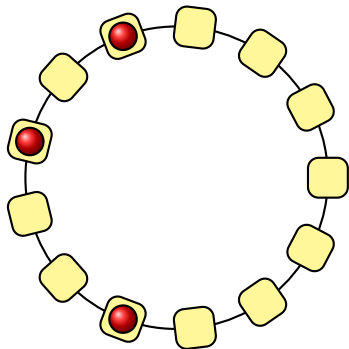
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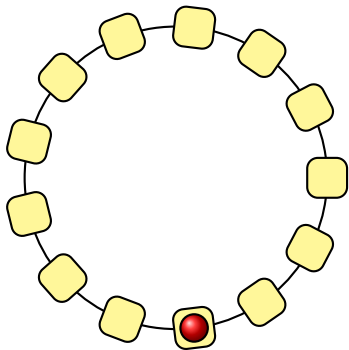
The synchronous version is standard.

The probability of passing is usually chosen as $1/2$,
but is a parameter in our analysis.

We study both versions (asynchronous and synchronous).

Self-Stabilization

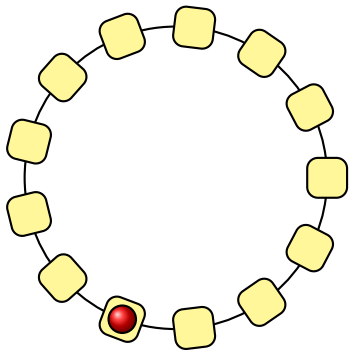
Self-stabilization: a concept of fault-tolerance:
Any configuration **leads to a "legitimate"** configuration
(in Herman: a single-token configuration).



Dijkstra (1974): In a token ring
there is **no symmetric deterministic** self-stabilizing algorithm.

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Time to Stabilization

Let $\mathbf{T} :=$ time until a 1-token configuration is reached.

What is $\mathbb{E}\mathbf{T}$?

Markov Chain argument $\Rightarrow \mathbb{E}\mathbf{T}$ is finite for any initial conf.

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showed $\mathbb{E}\mathbf{T} \leq 0.5N^2 \log N$
mentions improvement to $\mathbb{E}\mathbf{T} \leq c N^2$
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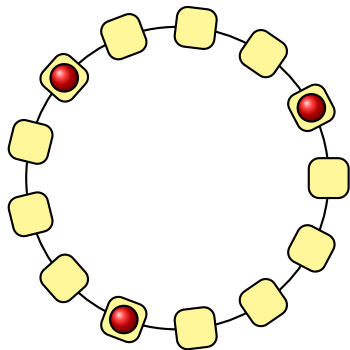
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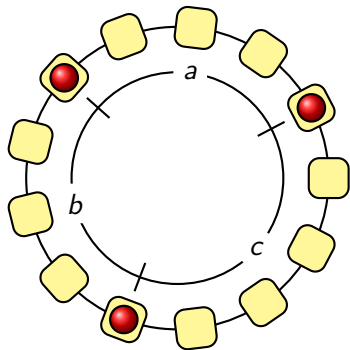
Then
$$\mathbb{E}\mathbf{T} \leq \left(\frac{\pi^2}{8} - \frac{29}{27} \right) \cdot \frac{N^2}{D}.$$

In particular, for synchronous and $r = 1/2$, we have $\mathbb{E}\mathbf{T} \leq 0.64N^2$.

The 3-Token Case



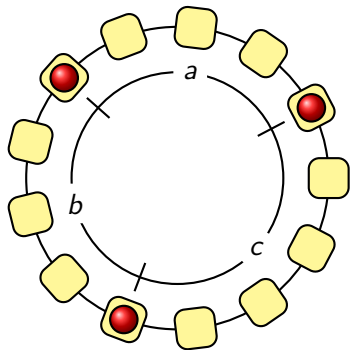
The 3-Token Case



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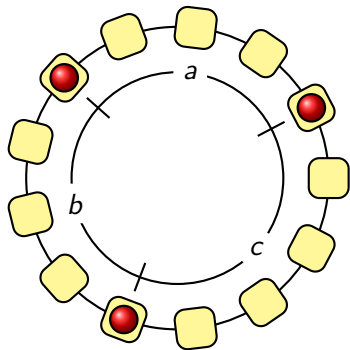
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Maximized for **equilateral** configuration with $a = b = c = N/3$

$$\Rightarrow \mathbb{E}\mathbf{T} = \frac{4}{27}N^2 = 0.148N^2$$

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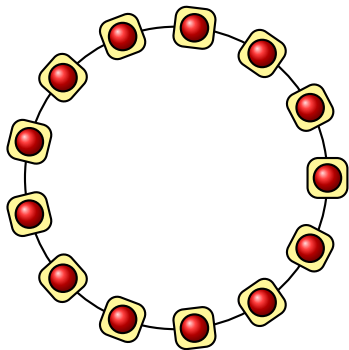
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Open conjecture [MM'05]:

3-token equilateral conf. is worse than **any** other conf.

If true, then $\mathbb{E}\mathbf{T} \leq 0.148N^2$ (cf. $\mathbb{E}\mathbf{T} \leq 0.64N^2$).

The Full Configuration



Theorem

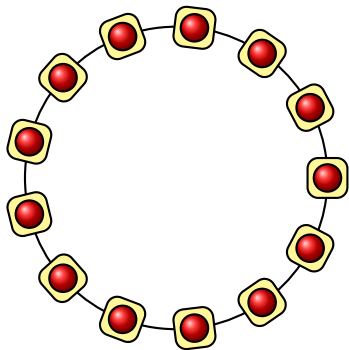
Let $D = r(1 - r)$ (for synchronous) or $D = \lambda$ (for asynchronous). Starting in the full configuration, for almost all odd N :

$$\mathbb{E}T \leq 0.0285N^2/D.$$

In particular, for synchronous and $r = 1/2$, we have $\mathbb{E}T \leq 0.114N^2$.

(cf. for **general conf.:** $\mathbb{E}T \leq 0.64N^2$
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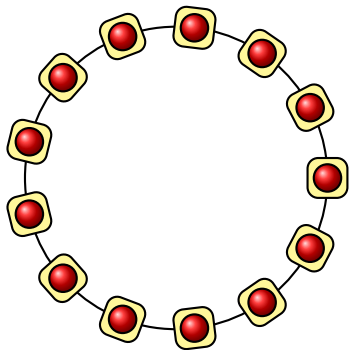
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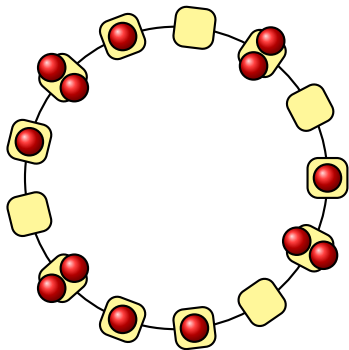
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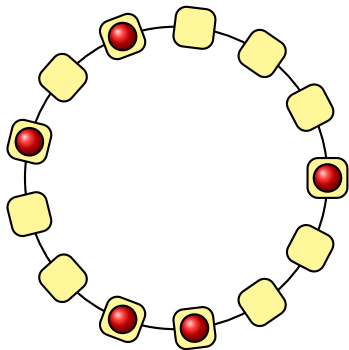
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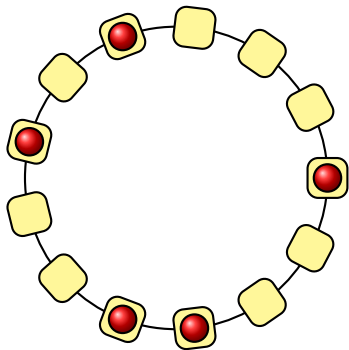
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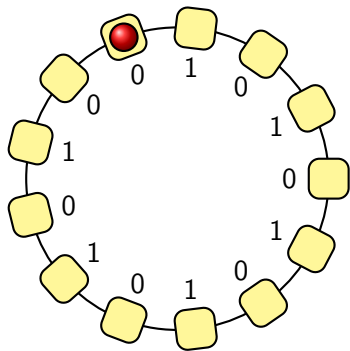
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⇒ no need for a different analysis

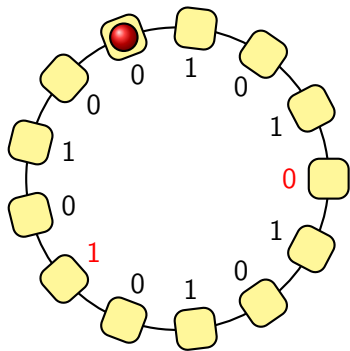
Restabilization

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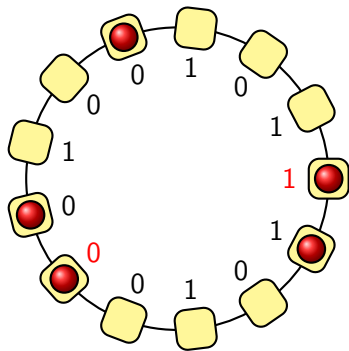
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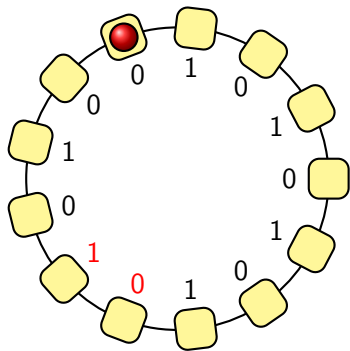
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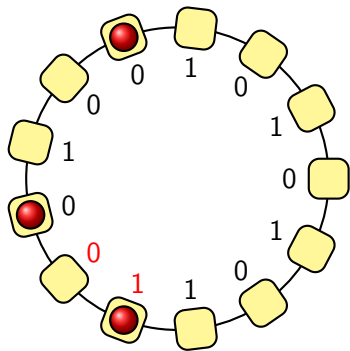
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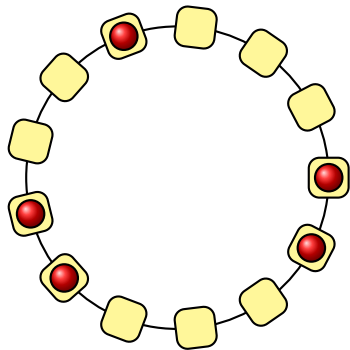
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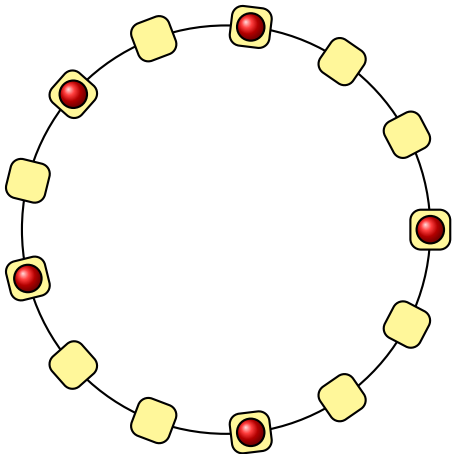
Theorem

Consider the synchronous protocol with $r \in (0.12, 0.88)$. Fix m .
Then for any flip- m configuration: $\mathbb{E}T = O(N)$

(cf. $O(N^2)$ for all other bounds)

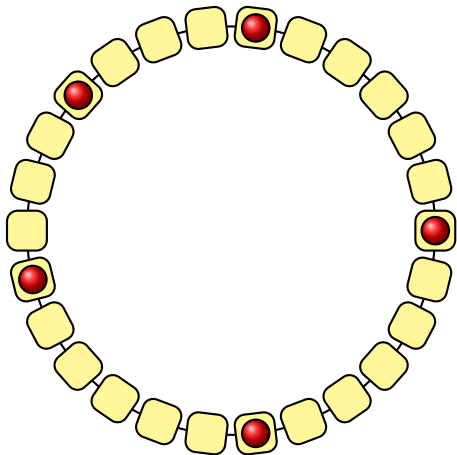
Interacting particles under random motion occur in:

- statistical mechanics
- neural networks
- spread of infections
- tumor growth
- ... (other physical and medical sciences)
- physical chemistry [[Balding'88](#)]



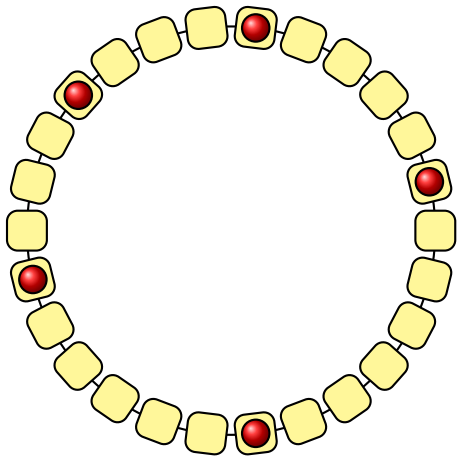
Key ingredients:

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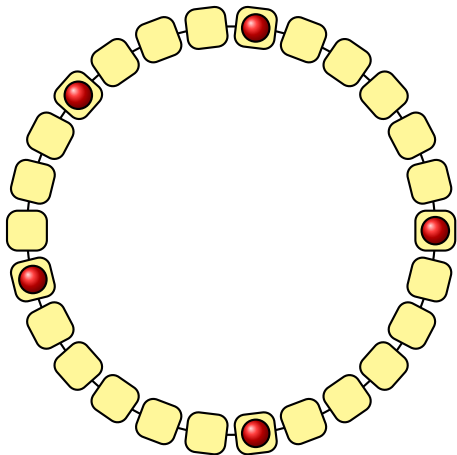
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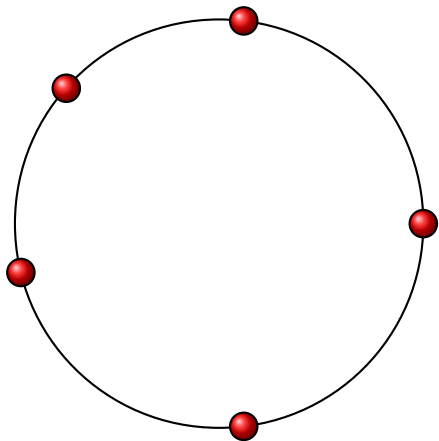
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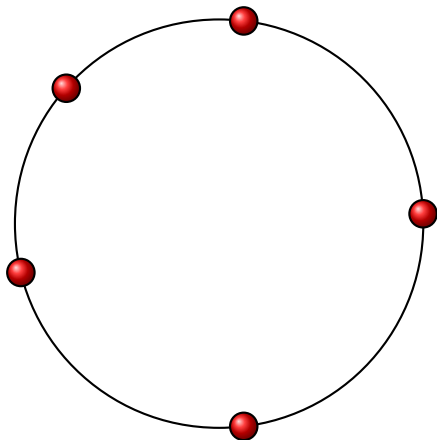
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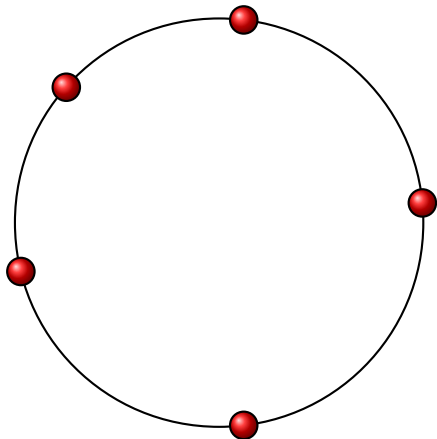
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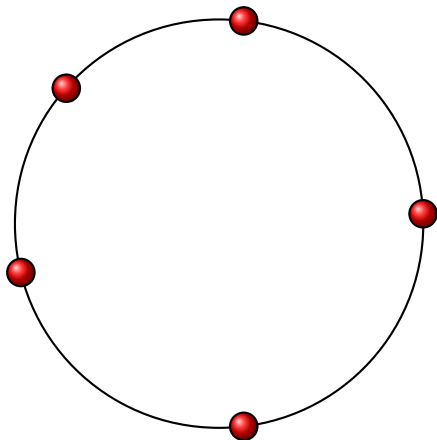
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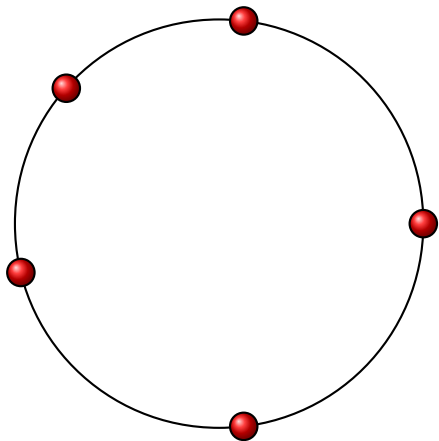
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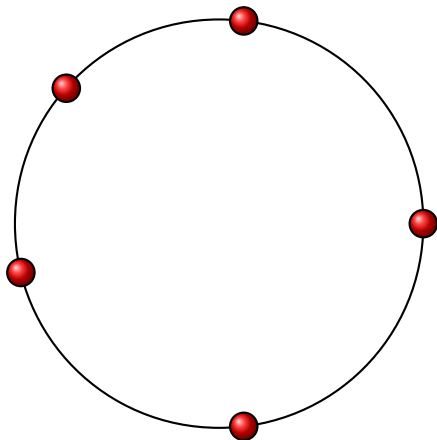
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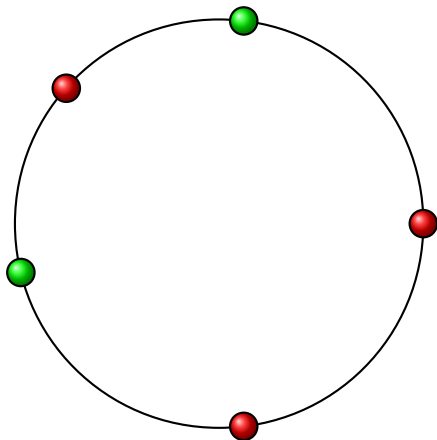
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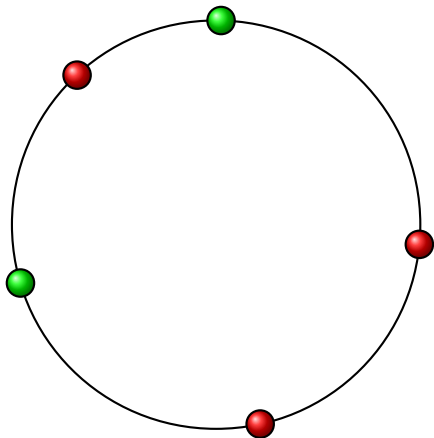
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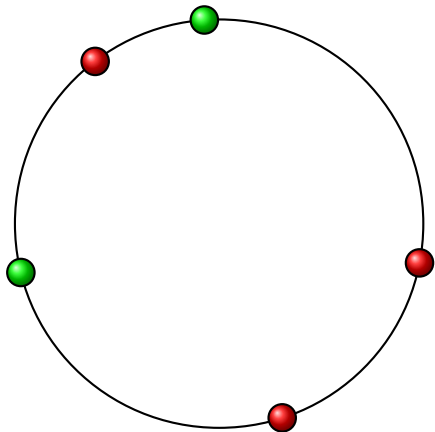
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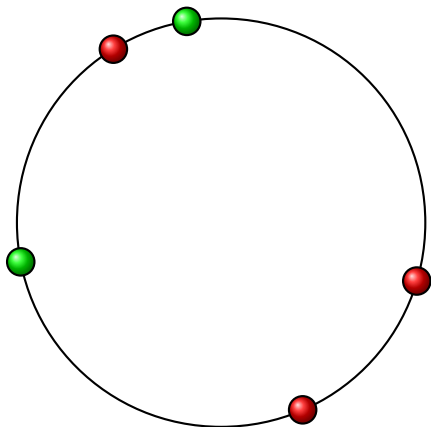
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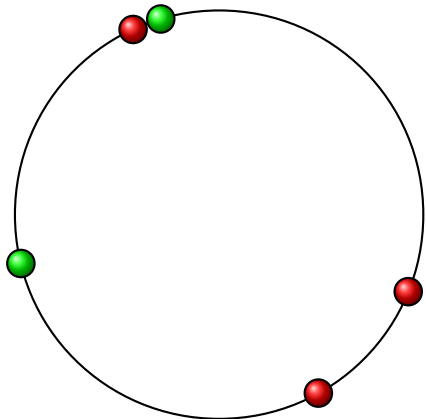
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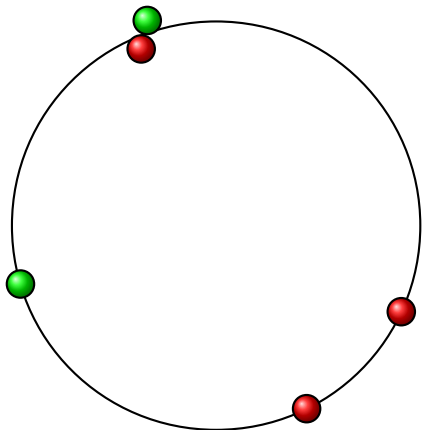
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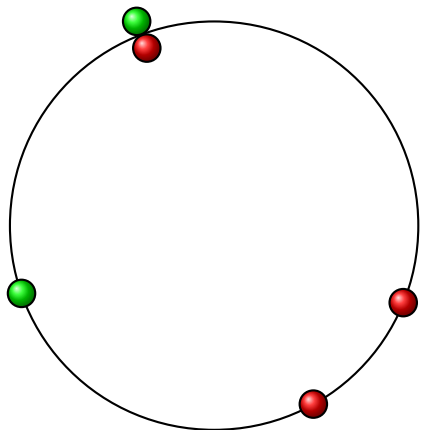
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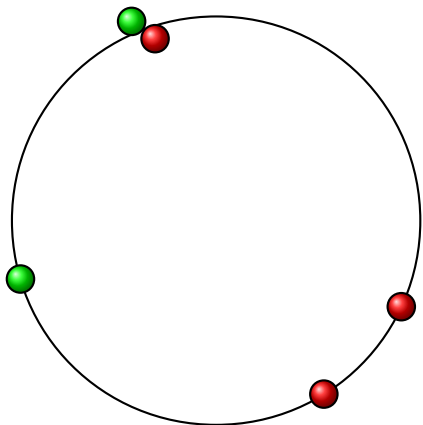
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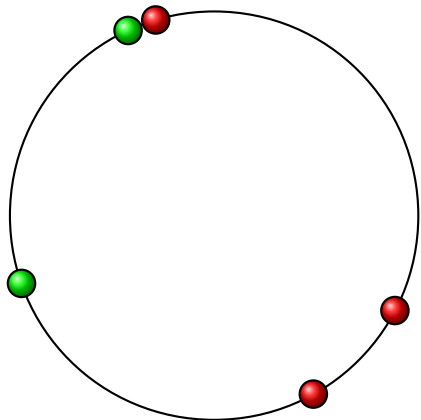
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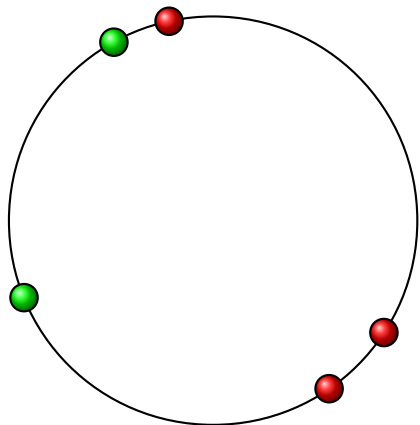
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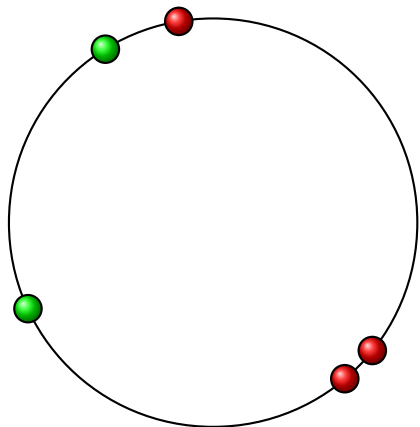
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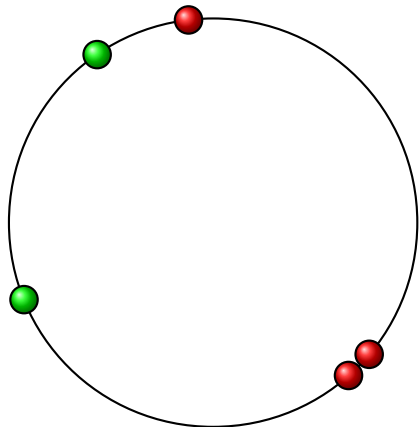
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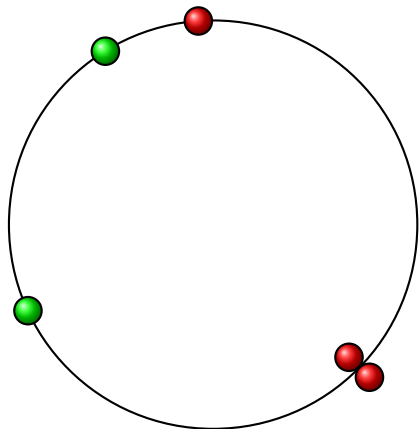
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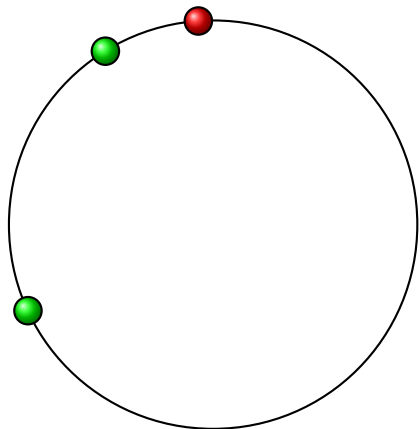
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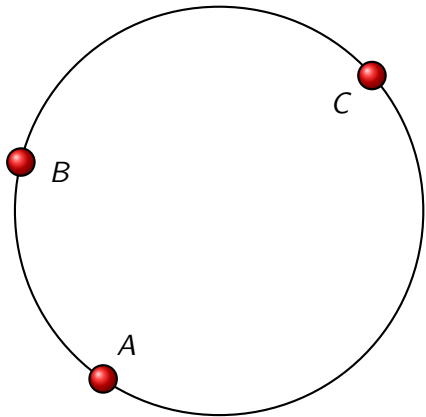
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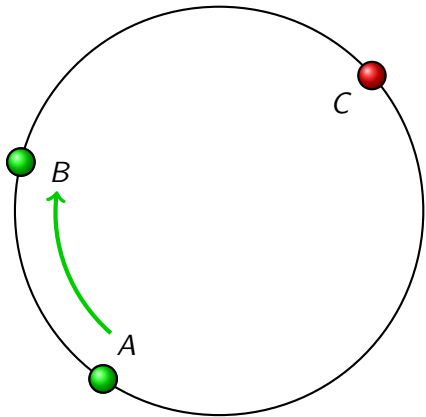
Two Colours: 3 Tokens



For any time $t \geq 0$:

$$\begin{aligned} \Pr(\mathbf{T} \leq t) &= \Pr(A \rightarrow B) - \Pr(B \rightarrow A) \\ &\quad + \Pr(B \rightarrow C) - \Pr(C \rightarrow B) \\ &\quad + \Pr(C \rightarrow A) - \Pr(A \rightarrow C) \end{aligned}$$

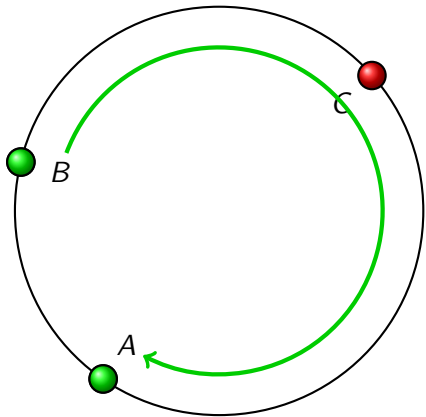
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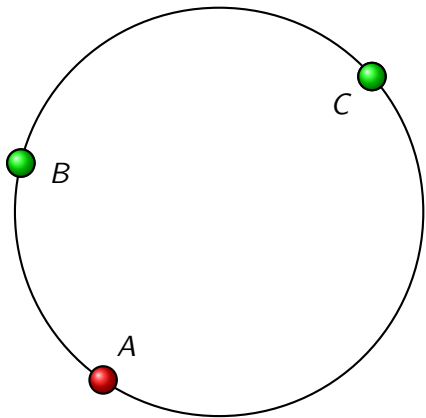
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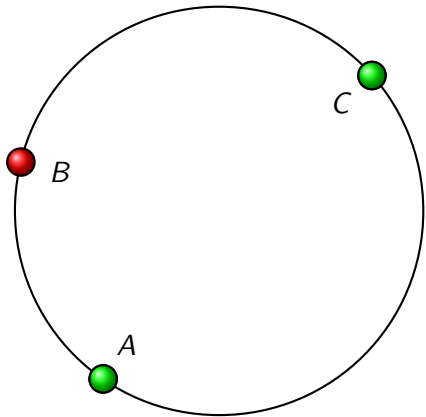
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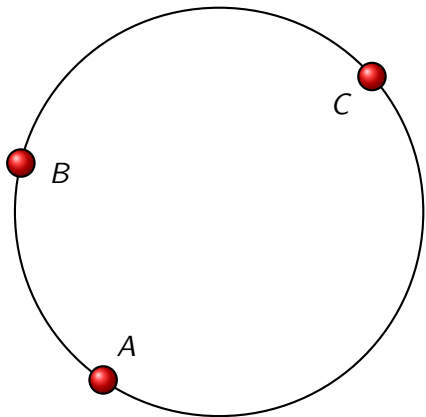
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$\Pr(\mathbf{T} \leq t)$ in terms of 1-D random walk with two absorbing barriers

- distributions well-known

Principle generalizes to more than 3 tokens.

We use Mathematics ...

with

$$e_0 = \frac{e_1^{-(1-d_0)}}{r \cdot (1-r) \cdot \pi^{k+2}} \cdot j_1 \cdots j_k \cdot (j_1^2 + \cdots + j_k^2) \quad \text{and}$$

$$e_i = j_1 \cdots j_k \cdot E_k((j^2)^{i+2/2}) \quad \text{for } i = 0, 2, \dots$$

Since $e_0 > 0$, the power series $e_0 + e_2 u^2 + e_4 u^4 + \cdots$ can be inverted. The inversion formula yields

$$\bar{f}_k(j; u) = a_0 + a_2 u^2 + a_4 u^4 + \cdots$$

with

$$a_0 = \frac{1}{e_0} = \frac{2^k}{r \cdot (1-r) \cdot \pi^{k+2} \cdot j_1 \cdots j_k \cdot (j_1^2 + \cdots + j_k^2)} \quad \text{and}$$

$$a_i = -a_0 \cdot \sum_{\ell=0, 2, \dots, i-2} a_\ell e_{i-\ell} \quad \text{for } i = 2, 4, \dots$$

It follows by an easy induction that

$$a_i = \frac{E_k((j^2)^i)}{j_1 \cdots j_k \cdot (j_1^2 + \cdots + j_k^2)^{i+2/2}} \quad \text{for } i = 0, 2, \dots$$

Using further values of the Taylor coefficients c_i, d_i from above, a straightforward but tedious computation shows that

$$a_4 = \frac{2^{k-4} P(j)}{45 \cdot \pi^{k-2} \cdot r \cdot (1-r) \cdot j_1 \cdots j_k \cdot (j_1^2 + \cdots + j_k^2)^3},$$

where

$$P(j) = \sum_{1 \leq i_1 < i_2} -3j_{i_1}^6 + \sum_{1 \leq i_1 < i_2 \leq k} -9(j_{i_1}^6 j_{i_2}^2 + j_{i_1}^2 j_{i_2}^6) +$$

$$\sum_{1 \leq i_1 < i_2 \leq k} (720(r(1-r))^2 - 240r(1-r) + 8) j_{i_1}^4 j_{i_2}^4 +$$

$$\sum_{1 \leq i_1 < i_2 < i_3 \leq k} (720(r(1-r))^2 - 300r(1-r) + 13) (j_{i_1}^4 j_{i_2}^2 j_{i_3}^2 + j_{i_1}^2 j_{i_2}^4 j_{i_3}^2 + j_{i_1}^2 j_{i_2}^2 j_{i_3}^4) +$$

$$\sum_{1 \leq i_1 < i_2 < i_3 < i_4 \leq k} (1440(r(1-r))^2 - 720r(1-r) + 60) j_{i_1}^2 j_{i_2}^2 j_{i_3}^2 j_{i_4}^2.$$

We have determined the above coefficients of $P(j)$ using the computer algebra system Maple. Now it is straightforward to verify that all coefficients of $P(j)$ are negative, if $\frac{3-2\sqrt{2}}{12} < r \cdot (1-r) \leq \frac{1}{4}$. Those inequalities hold, if $r \in (\frac{1}{2} - \frac{\sqrt{27}}{6}, \frac{1}{2} + \frac{\sqrt{27}}{6})$. \square

The following lemma is used as an induction step in the proof of Lemma 10 below.

Lemma 13. If $k \in \{2, 3, \dots\}$ and $u > 0$, then

$$\lim_{j_k \rightarrow 0} (j_k \cdot \bar{f}_k(j_1, \dots, j_k; u)) = \frac{2}{\pi} \bar{f}_{k-1}(j_1, \dots, j_{k-1}; u),$$

where the j_i vary over the nonnegative reals. Consequently, with the Taylor expansion $\bar{f}_k(j; u) = a_{k,0}(j) + a_{k,2}(j)u^2 + a_{k,4}(j)u^4 + \cdots$ from Lemma 12 we also have

$$\lim_{j_k \rightarrow 0} (j_k \cdot a_{k,i}(j_1, \dots, j_k)) = \frac{2}{\pi} a_{k-1,i}(j_1, \dots, j_{k-1}).$$

Proof. As $h(0, u) = 1$, it suffices to show that $\lim_{j_k \rightarrow 0} \frac{j_k \sin(j_k \pi u)}{1 - \cos(j_k \pi u)} = \frac{2}{\pi}$. This follows easily from l'Hopital's rule:

$$\begin{aligned} \lim_{j_k \rightarrow 0} \frac{j_k \sin(j_k \pi u)}{1 - \cos(j_k \pi u)} &= \lim_{j_k \rightarrow 0} \frac{\sin(j_k \pi u) + j_k \cos(j_k \pi u) \cdot \pi u}{\sin(j_k \pi u) \cdot \pi u} \\ &= \frac{1}{\pi u} + \lim_{j_k \rightarrow 0} \frac{j_k \cos(j_k \pi u)}{\sin(j_k \pi u)} \\ &= \frac{1}{\pi u} + \lim_{j_k \rightarrow 0} \frac{\cos(j_k \pi u) - j_k \sin(j_k \pi u)}{\cos(j_k \pi u) \cdot \pi u} \\ &= \frac{1}{\pi u} + \frac{1}{\pi u} = \frac{2}{\pi u} \end{aligned}$$

\square

Now we can prove Lemma 10 which is restated here.

Lemma 10. For any fixed $k \in \mathbb{N}_+$ and $r \in (\frac{1}{2} - \frac{\sqrt{27}}{6}, \frac{1}{2} + \frac{\sqrt{27}}{6})$, we have

$$\sum_{j \in \{1, \dots, N-1\}^k} |\bar{f}_k(j; y; 1/N) - a_0(j)| = O\left(\frac{(\log N)^k}{N^2}\right).$$

Proof. By Lemma 12 it is equivalent to prove

$$\sum_{j \in \{1, \dots, N-1\}^k} \left| \frac{a_2(j)}{N^2} + \frac{a_4(j)}{N^4} + \cdots \right| = O\left(\frac{(\log N)^k}{N^2}\right).$$

Notice that an easy induction shows that

$$\sum_{j \in \{1, \dots, N-1\}^k} \frac{1}{j_1 \cdots j_k} = \sum_{j_k=1}^{N-1} \frac{1}{j_k} \sum_{j \in \{1, \dots, N-1\}^{k-1}} \frac{1}{j_1 \cdots j_{k-1}} = O((\log N)^k).$$

Hence, with Lemma 12 we have

$$\sum_{j \in \{1, \dots, N-1\}^k} \left| \frac{a_2(j)}{N^2} \right| = \frac{1}{N^2} \cdot \sum_{j \in \{1, \dots, N-1\}^k} \frac{O(1)}{j_1 \cdots j_k} = O\left(\frac{(\log N)^k}{N^2}\right).$$

Summary

Upper bounds for $\mathbb{E}T$ for various initial configurations:

- arbitrary configuration (improved bound)
- full configuration (new bound)
- random configuration (new bound)
- flip- m configurations (new bound, $O(N)$)

Also considered:

- asynchronous version
- synchronous version with passing prob. $\neq 1/2$

Main techniques used:

- limit \rightarrow Brownian motion
- two colours

Main open question is still:

- Is the 3-token equilateral conf. the worst case?

Particle Collision: CERN

