

Efficient Analysis of Probabilistic Programs with an Unbounded Counter

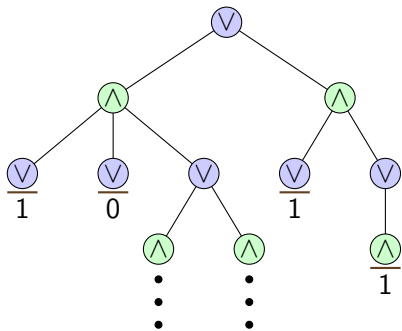
CAV 2011

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¹Masaryk University, Brno, Czech Republic

²University of Oxford, UK

Evaluation of And-Or Trees



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procedure AND(node)
  if node is a leaf
    return node.value
  else
    for each successor s of node
      if OR(s) = 0 then return 0
    return 1
```

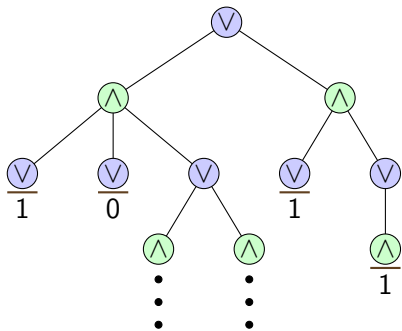
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- cannot tell: program may not even terminate

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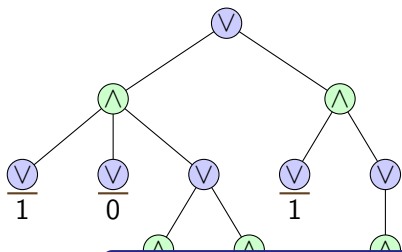
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⇒ probabilistic assumptions:

- AND node has 3 kids in average (geom. distribution)
- OR node has 2 kids in average
- a branch has length 4 in average
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Under these probabilistic assumptions:

Approximate **efficiently** the **expected runtime**

sary)

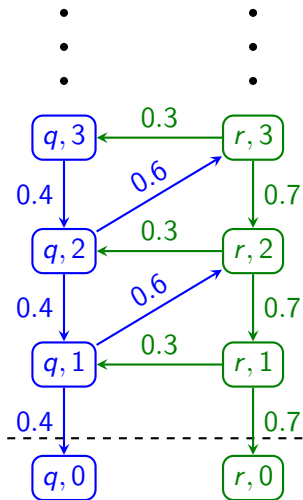
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Probabilistic Counter Machines

Probabilistic Counter Machines induce infinite Markov chains:

$$\begin{array}{ll} q \xrightarrow{0.6} r(+1) & r \xrightarrow{0.3} q(\pm 0) \\ q \xrightarrow{0.4} q(-1) & r \xrightarrow{0.7} r(-1) \end{array}$$



Modeling a Program as Prob. Counter Machine

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procedure AND(node)
  if node is a leaf
    return node.value
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```

if leaf, return 0 or 1 :

$$\text{and} \xrightarrow{\ell \cdot z} \text{and0}(-1)$$

$$\text{and} \xrightarrow{\ell \cdot (1-z)} \text{and1}(-1)$$

otherwise, call OR :

$$\text{and} \xrightarrow{1-\ell} \text{or}(+1)$$

if OR returns 0, return 0 immediately :

$$\text{or0} \xrightarrow{1} \text{and0}(-1)$$

otherwise, maybe call another OR :

$$\text{or1} \xrightarrow{x} \text{or}(+1)$$

$$\text{or1} \xrightarrow{1-x} \text{and1}(-1)$$

Applications of Probabilistic Counter Machines

PCMs model **infinite-state** probabilistic programs

- recursion
- unbounded data structures

PCMs = discrete-time **Quasi-Birth-Death** processes

- well established stochastic model
 - studied since the late 60s
- ⇒ queueing theory, performance evaluation, ...

Recently: **Games** over (Probabilistic) Counter Machines

- energy games [Chatterjee, Doyen et al.]
- ⇒ optimizing resource consumption in portable devices

Related Model: Probabilistic Pushdown System

Probabilistic **Pushdown Systems** modify a stack:

$$q(X) \xrightarrow{0.3} r(YY)$$

$$q(X) \xrightarrow{0.5} r(X) \quad q(Y) \hookrightarrow \dots \quad r(X) \hookrightarrow \dots \quad r(Y) \hookrightarrow \dots$$

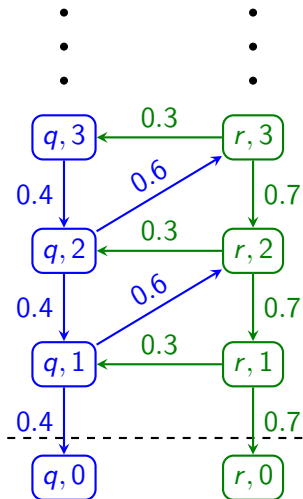
$$q(X) \xrightarrow{0.2} q(\varepsilon)$$

Prob. Pushdown Systems (equivalently, Recursive Markov Chains) are more general, but more expensive to analyze.

PCMs are Prob. Pushdown Systems with a **single stack symbol**.

Probabilistic Counter Machines

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Trend

Runtime $T :=$ number of steps from $(q, 1)$ to $(*, 0)$

We want to efficiently approximate $\mathbb{E}T$.

Trend $t :=$ “average increase of the counter per step”

Assume $t < 0$.

Intuition: The more negative the trend t , the smaller T .

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Proposition (from martingale theory: Azuma's inequality)

Let $m^{(0)}, m^{(1)}, m^{(2)}, \dots$ be random variables with $m^{(0)} = 1$.

Let $t < 0$.

Assume $\mathbb{E}(m^{(k+1)} \mid m^{(k)}) = m^{(k)} + t$ for all k .

Then for all k : $\Pr(m^{(k)} \geq 1) \leq a^k$, where $a = e^{-t^2/2} < 1$.



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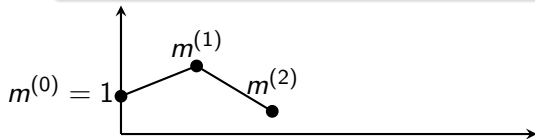
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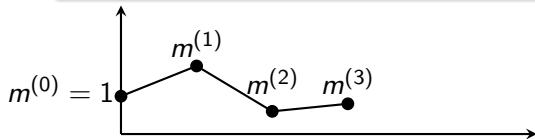
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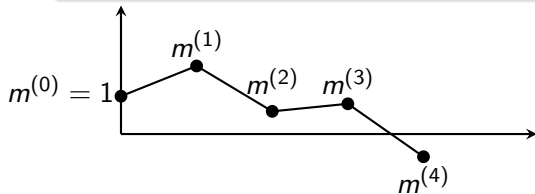
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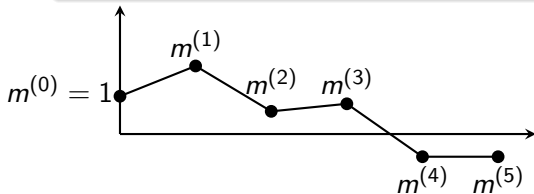
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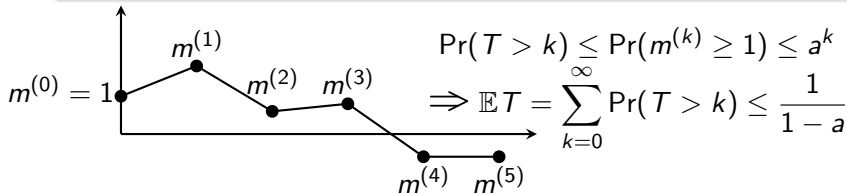
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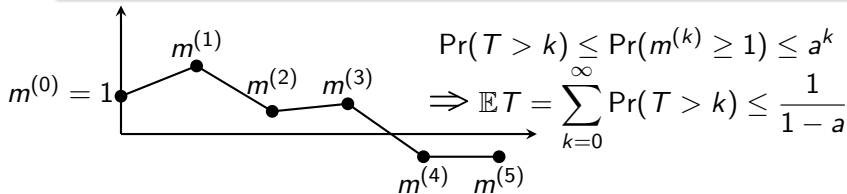
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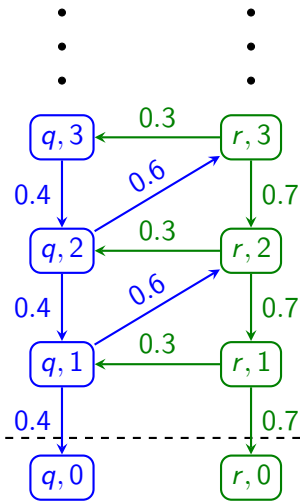
Let $t < 0$. **But the trend must be independent of k** :-)

Assume $\mathbb{E}(m^{(k+1)} \mid m^{(k)}) = m^{(k)} + t$ for all k .

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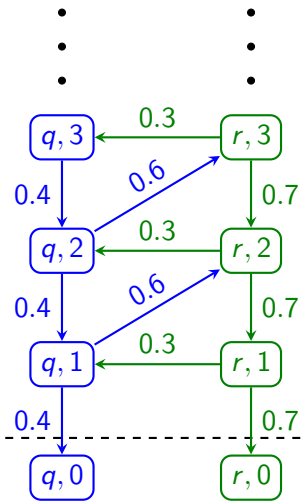
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Average counter increase depends on state:

$$\begin{pmatrix} 0.4 \cdot (-1) + 0.6 \cdot (+1) \\ 0.3 \cdot 0 + 0.7 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0.2 \\ -0.7 \end{pmatrix}$$

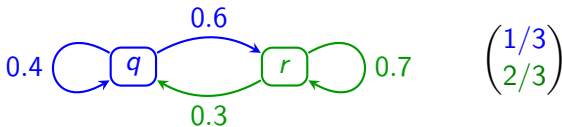
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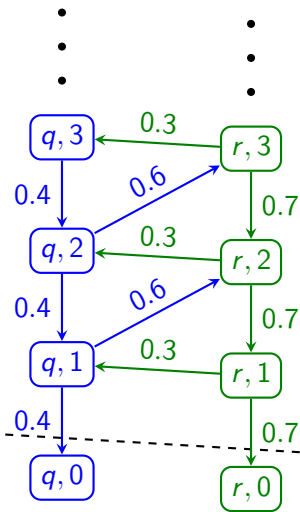
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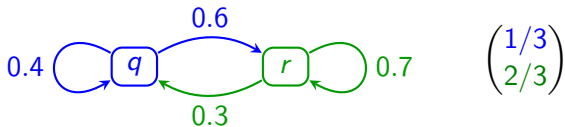
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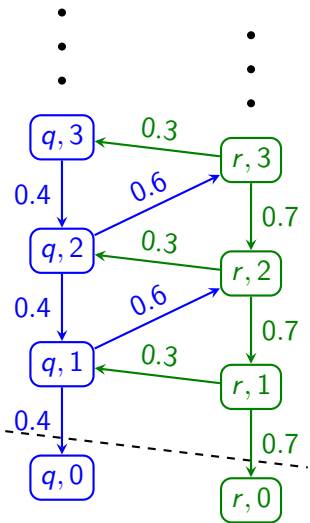
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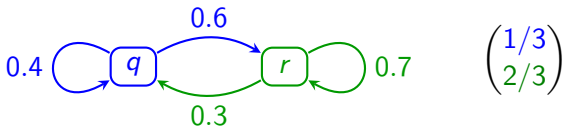
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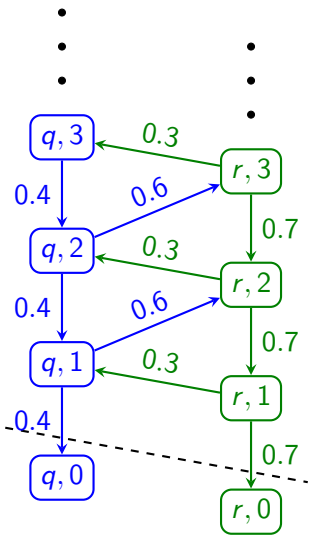
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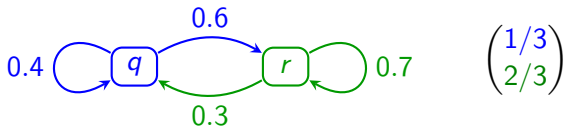
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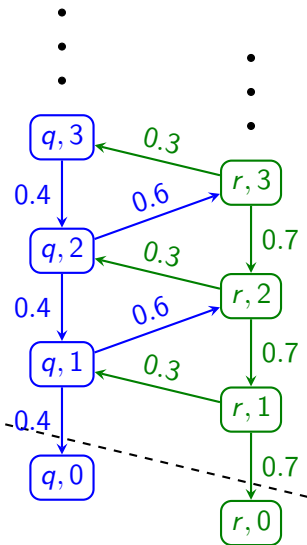
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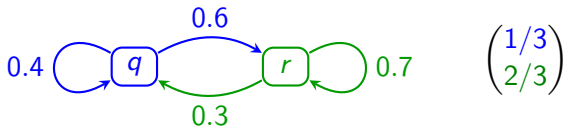
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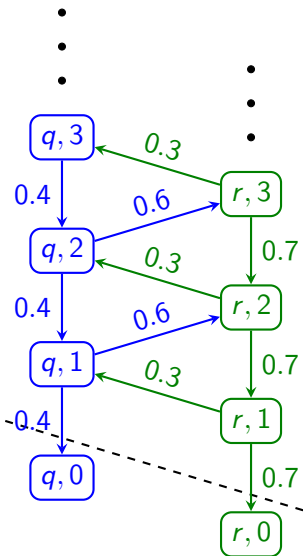
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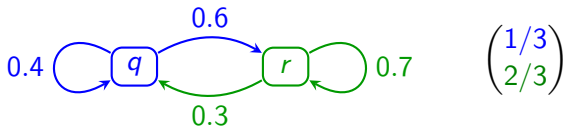
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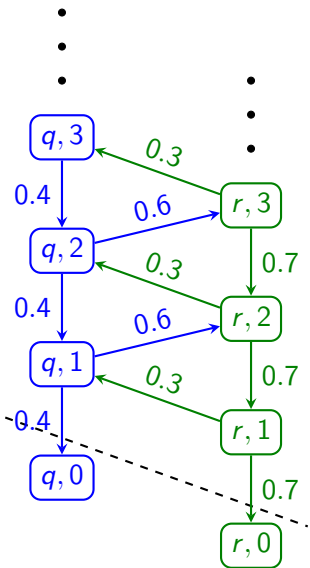
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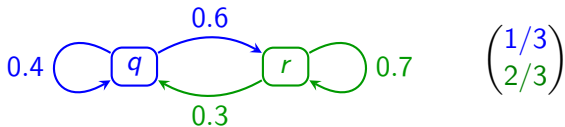
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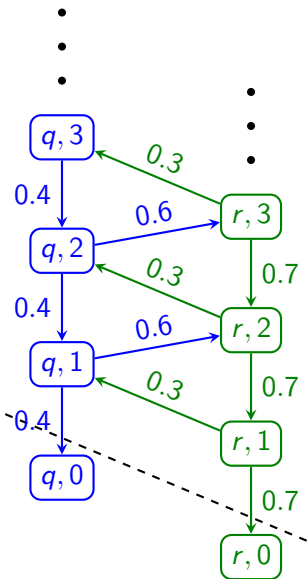
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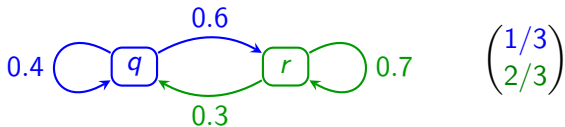
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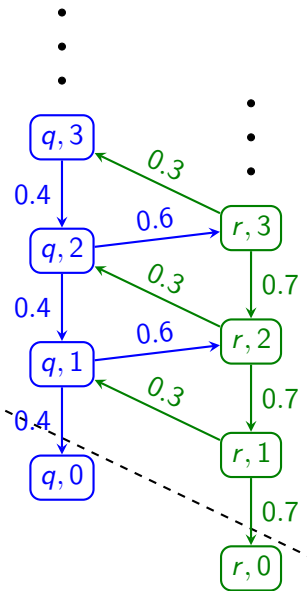
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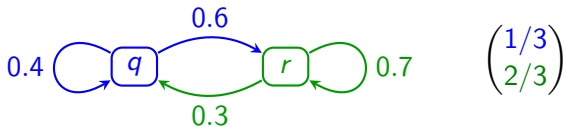
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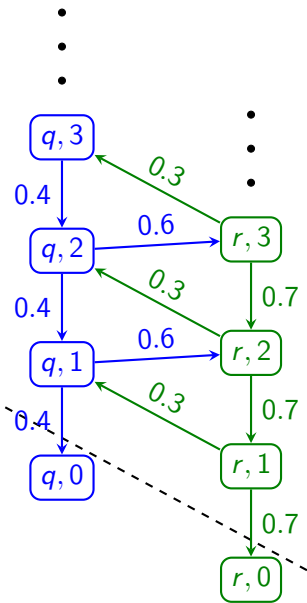
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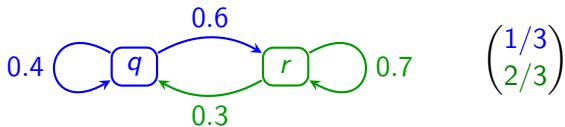
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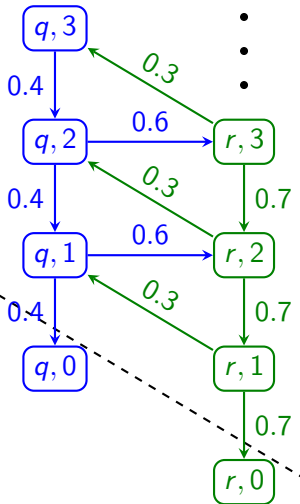
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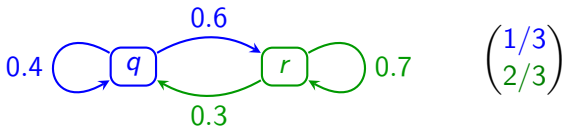
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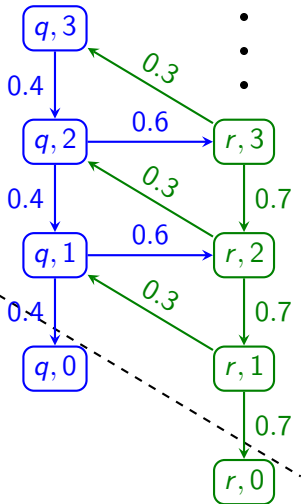
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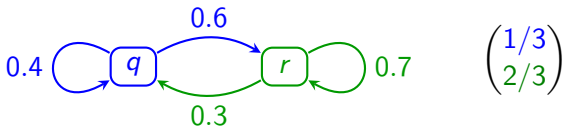
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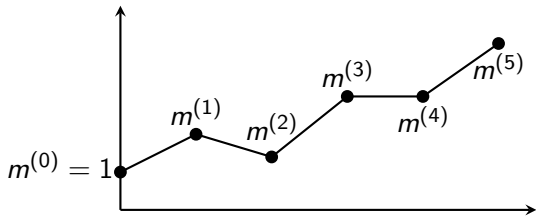
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\Rightarrow expected height increase: $t = -0.4$.
independent of control state :-)

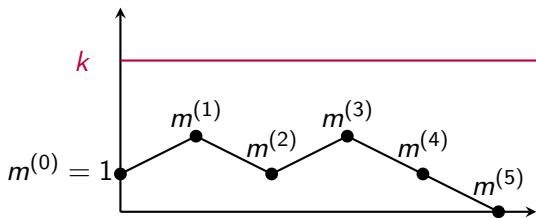
Positive Trend



If $t > 0$, then $\Pr(T = \infty) > 0$.

$\mathbb{E}(T \mid \text{finite})$ can be bounded as before.

Zero Trend



Proposition (from martingale theory: Optional stopping theorem)

Let $m^{(0)}, m^{(1)}, m^{(2)}, \dots$ be random variables with $m^{(0)} = 1$.

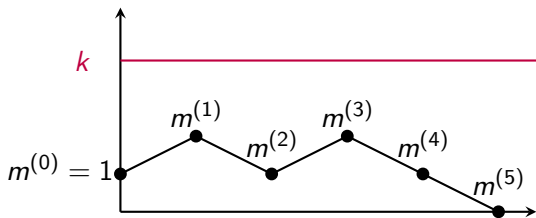
Assume $\mathbb{E}(m^{(i+1)} \mid m^{(i)}) = m^{(i)}$ for all i .

Let $k \in \mathbb{N}$.

Let τ be the first time with $m^{(\tau)} \notin (0, k)$.

Then $\mathbb{E}m^{(\tau)} = 1$.

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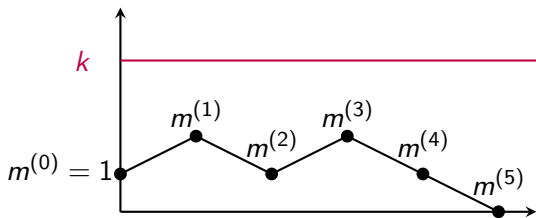
Then $\mathbb{E}m^{(\tau)} = 1$.

Assuming all jumps are $+1, \pm 0, -1$, we must have

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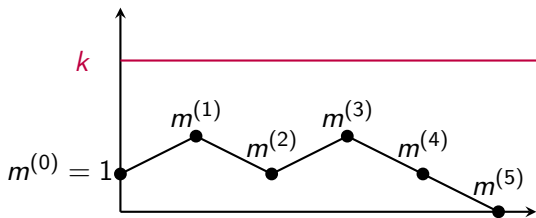
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$\Rightarrow \Pr(T \geq k) \geq \Pr(m^{(\tau)} = k) = 1/k$ and hence $\mathbb{E}T = \infty$

Finiteness of Expected Time

We condition on runs $q \downarrow r$: from $(q, 1)$ reach $(r, 0)$
(e.g., consider $[and \downarrow and0]$, $[and \downarrow and1]$)

Theorem

Either some easy case holds or one of the following:

- *If trend $t \neq 0$, then $\mathbb{E}(T \mid q \downarrow r) \leq 85000 \cdot \frac{|Q|^6}{x_{\min}^{5|Q|+|Q|^3} \cdot t^4}$.*
- *If trend $t = 0$, then $\mathbb{E}(T \mid q \downarrow r)$ is infinite.*

Corollary

Whether $\mathbb{E}(T \mid q \downarrow r)$ is finite can be decided in polynomial time.

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Corollary

Whether $\mathbb{E}(T \mid q \downarrow r)$ is finite can be decided in polynomial time.

But we want an **approximation** of $\mathbb{E}(T \mid q \downarrow r)$.

“return probabilities” :

$$[q \downarrow r] := \Pr(\text{from } (q, 1) \text{ reach } (r, 0))$$

Proposition (from [EWY'08])

- If $[q \downarrow r] > 0$, then $[q \downarrow r] \geq x_{\min}^{|Q|^3}$.
- $[q \downarrow r]$ can be approximated within any error $\varepsilon > 0$ in time $\text{poly}(|\mathcal{S}|, \log(1/\varepsilon))$ in unit-cost arithmetic.

(does not hold for pushdown systems)

Approximating Expected Runtime

Theorem

The value $\mathbb{E}(T \mid q \downarrow r)$ can be approximated within any error $\varepsilon > 0$ in time $\text{poly}(|\mathcal{S}|, \log(1/\varepsilon))$ in unit-cost arithmetic.

Use the following procedure:

- Set up an equation system $A\mathbf{x} = \mathbf{1}$. (system already known)
Solution vector contains $\mathbb{E}(T \mid q \downarrow r)$ for all $q, r \in Q$.
The matrix A contains return probabilities.
- Approximate A by approximating the return probabilities.
- Solve the approximated equation system.

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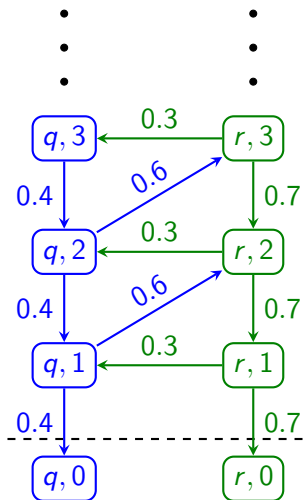
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Precision of this method depends on the **condition number** of A .

The condition number is good enough as:

- the return probabilities cannot be too small
- the solution cannot be too large (by our bound on $\mathbb{E}T$)

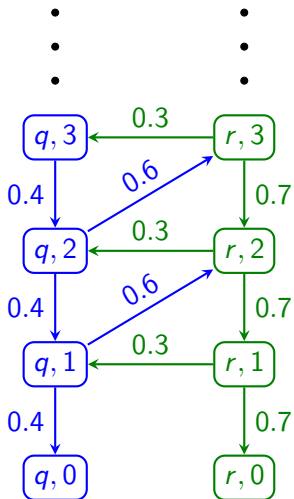
Rules for Zero Counter



Now allow rules for zero counter
(not -1)

\Rightarrow all runs are infinite

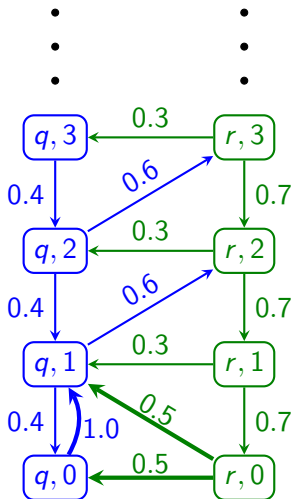
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Theorem

Given an ω -regular specification in terms of a Rabin automaton \mathcal{R} , the probability of a run satisfying the specification can be approximated within any error $\varepsilon > 0$ in time $\text{poly}(|\mathcal{S}|, |\mathcal{R}|, \log(1/\varepsilon))$ in unit-cost arithmetic.

Proof uses again “trend”-based martingale arguments.

Summary

- Probabilistic Counter Machines model infinite-state systems with a regular “counter-like” structure.
- Expected runtime and other quantities can be efficiently approximated (cf. prob. pushdown systems).
- Martingale techniques play a key role for the analysis.

Thank you!